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WAVE MECHANICS

BEING ONE ASPECT OF THE NEW QUANTUM THEORY

BY

H. T. FLINT, Ph.D., D.Sc.

READER IN PHYSICS IN THE UNIVERSITY OF LONDON

WITH 19 DIAGRAMS

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GENERAL PREFACE

THIS series of small monographs is one which should commend itself to a wide field of readers.

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From a consideration of the list of authors it is clear that the reader need have no doubt of the accuracy of the general accounts found in these volumes.

O. W. RICHARDSON

KING'S COLLEGE
June, 1928

PREFACE

THIS little book is written in the hope that it will be of assistance to those who are interested in the most recent development of physics; the new method of approach to atomic problems.

The aim has been to present to the reader in a reasonably simple manner an account of the theory, with its relation to existing physical theory, as developed by de Broglie and Shroedinger. This theory is properly called Wave Mechanics and it is based on conceptions of continuity. It is in fact a classical theory, for the methods adopted are those which were familiar to physicists before the beginning of this century. The mathematical analysis required has been extensively developed by earlier generations of physicists and may be regarded as part of a familiar equipment. For this reason the theory is approached by ways that the physicist of to-day may not have explored before but which are well mapped out and amply provided with sign-posts.

No attempt has been made to describe more than one of the two methods of studying these problems. The second has originated with Bohr, Dirac and Heisenberg and is fundamentally different in character for it is based on a conception of discontinuity and is called The New Quantum Mechanics. The mathematical analysis in this case is less familiar to physicists, is less developed and some of it is new.

We do not wish to appear to regard one view as more important or more powerful in dealing with these problems than the other. It is too early in the history

of the new development to take sides, and to some extent preference for the one or the other view is a matter of taste. It is remarkable that two views so widely different should lead to the same results and though the reason for this may be easily explained by a comparison of the notations employed, we must not lose sight of the fundamental difference in physical character.

The list of references to original and other works printed at the end of the book is given in the hope that it will be of help to those who wish to consult original papers and also as an acknowledgment of the sources which have been drawn upon in writing the book. In addition we must acknowledge also two lectures by Prof. E. T. Whittaker given in the course of the last year.

H. T. F.

Wheatstone Laboratory
King's College
London

December, 1928

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WAVE MECHANICS

INTRODUCTION

UNTIL the beginning of the present century the development of theoretical physics had progressed so far and so successfully that the principles underlying natural phenomena seemed to be understood and it was expected that the same principles would hold in any new domain of knowledge which might be discovered. Nowadays after nearly thirty years of further research we are familiar with the inadequacy of the wonderful structure which previous generations have built up and which we know as the classical theory. It will be worth while to consider briefly the development of this theory and to glance at its far-reaching successes before pointing out where it proved inadequate.

It is well known that Newton was the first to work out the laws of dynamics and this branch of knowledge became a self-contained domain as the result of his work and that of his successors. His law of gravitation immediately opened up a new field for the newly established science in which it reaped, and still reaps, success after success. In the eighteenth and nineteenth centuries in the hands of mathematicians, astronomers and physicists, mechanics attained such a degree of elegance that the whole science could be founded upon a single principle known as the principle of least action which was first developed by Maupertuis and later in a quite different way by Hamilton.

The principles of mechanics came to be applied extensively in hydrodynamics, in the theory of sound and in optics, and in each case with success. Later

on the development of statistical mechanics by Maxwell, Boltzmann and Gibbs offered an explanation of the second law of thermodynamics which at first appeared to stand aside from the mechanical theory.

The mechanical theory of optics lost ground and was finally defeated altogether by the wave theory, the reaction occurring about the beginning of the nineteenth century. Later in this century came the electromagnetic theory in which Maxwell combined the results of his predecessors into an exact theory and showed that optics was a branch of the theory of electromagnetism.

The century was distinguished also by the experimental work of Sir J. J. Thomson on the discontinuity of electricity and by the introduction of the hypothesis of discontinuity into the Maxwellian theory by H. A. Lorentz.

The discovery of positive and negative charges and of their universal occurrence led one school of thought to the electrical theory of matter. If matter is ultimately electricity then the phenomena of Nature may be expected to be ultimately electromagnetic and the forces experienced are electromagnetic in origin. One might anticipate that mechanics itself would turn out to be a branch of electromagnetism. The introduction of Lorentz's formula for the force on a moving charge to complete Maxwell's equations was a step to joining up the mechanical and electromagnetic domain.

The state of affairs at the beginning of this century certainly justified the hope that there would be an early blending of these apparently separate domains into a unified physical territory.

The opening years of the century, however, brought to light a very serious difficulty. It proved to be one of many which the classical theory had to encounter. The interest in the union of existing theories was lost in the attempt to account by known principles for the experimental results now clamouring for explanation.

There were originally two clouds darkening the classical sky, with one of which we are not now concerned. This was the difficulty associated with the

Michelson-Morley experiment which has been removed by the principle of relativity. This principle does not actually depart from the methods of classical physics, it embodies the same spirit and though it has brought an attitude to physical phenomena which is very far removed from that of our classical predecessors it reveals the older theory as a close approximation to the new. No matter how differently one may regard the law of gravitation in the light of the general principle of relativity we can still describe Newton's law of gravitation as a first approximation and in fact a very close approximation. Nevertheless it was natural to hope that the modification of Newtonian mechanics which came with the new theory would help to smooth away the difficulties.

The second difficulty indicated a breakdown in statistical mechanics. One of the consequences of this theory is the principle of equipartition of energy. This principle may be understood by considering the case of a molecule which moves without rotation in a collection of similar and similarly moving molecules. The kinetic energy corresponding to the x-co-ordinate of the particle is $\frac{1}{2}m\dot{x}^2$ where $\dot{x}$ denotes the velocity and m the mass of the particle. The principle of equipartition of energy states that the average value of $\frac{1}{2}m\dot{x}^2$ is the same as the average value of $\frac{1}{2}m\dot{y}^2$ and of $\frac{1}{2}m\dot{z}^2$ where y and z denote the other two co-ordinates. This value is in fact $\frac{1}{2}$RT where R is the gas constant and T the absolute temperature.

In this case we have three co-ordinates which fix the position of the particle; in general there may be any number to fix each unit of which the assembly is composed. The principle of equipartition states that the kinetic energy corresponding to each component has on the average the same value $\frac{1}{2}$RT. This result, which is an unavoidable conclusion from the classical theory, when applied to the problem of determining how the energy in radiation is distributed with respect to the wave lengths leads to a contradiction with experiment. Although it is in agreement with the

experimental findings in the case of long waves it is inaccurate in the case of short waves and leads to the result that the total energy density of radiation is infinite. This can have no physical meaning.

It was in attempting to explain on theoretical grounds this question concerning black body radiation that Planck introduced his quantum hypothesis which formed the starting-point of the old quantum theory. Another difficulty arose in that the classical theory of specific heats led to the law of Dulong and Petit that the atomic heats of the elements were constant. There were important deviations from this theory which only received an explanation by the use of quantum principles which were additions to the classical theory and which could in no way be deduced from it.

The study of photo-electric phenomena brought to light the difficulty that electrons emitted from metals by radiation incident upon them came away with an energy dependent on the frequency and not upon the intensity of the incident rays. This again received a theoretical explanation by Einstein by a principle again foreign to the classical theory.

In this way a theory grew up whose content appeared at variance with accepted views but which was applied to particular problems simultaneously with the classical theory and was therefore looked upon as an additional limitation to that theory.

The state of physics appeared to many who wrote on this subject about that time to be threatened with chaos since one set of laws appeared to work well in one domain and another quite different set in another domain. H. Poincaré shortly before his death and after the Solvay Congress of 1911 showed that it was necessary to introduce some quite new principle into theoretical physics, viz.: Planck's hypothesis.

The success of this new theory which we now call the old quantum theory was not yet so widespread as to make it acceptable to physicists in general, but its most brilliant success awaited it.

In 1913 Bohr published his atomic theory and so opened the most striking epoch in the history of spectra. The success of that theory, developed later by a generalization of his principles by Sommerfeld and W. Wilson, is well known. The application of Bohr's principles has turned the theory of spectra from being a collection of heterogeneous data into a well-ordered branch of physics and the success has been so extensive and at the same time so accurate in detail that the serious difficulties were often overlooked which have finally caused a revolt, in which Bohr himself has taken a leading part.

As a result of the old quantum theory a corpuscular character was again attributed to radiation, energy being supposed to be associated with it in the form of quanta of magnitude $h\nu$, where ν is the frequency of radiation and h is Planck's constant ($6{\cdot}55 \times 10^{-27}$ units). With these quanta we must associate a mass $\frac{h\nu}{c^2}$, according to the principle that energy has mass, which is a consequence of the theory of relativity.

One of the most striking applications of this is that of A. H. Compton, who considers the problem of the scattering of radiation by electrons as if it were a problem of impact of quanta and electrons. His results are in agreement with experiment. On the other hand, in the phenomena of interference we are compelled to hold to the wave theory of light and this is true even in the case of Röntgen rays as the crystal phenomena studied by Laue and Sir William and W. L. Bragg amply show.

These different phenomena show that radiation has apparently both a corpuscular and an extended character and the time has come to try and unite these apparently opposing aspects.

Two different lines of attack have been made upon these and other difficulties. One of them initiated by Heisenberg sought to describe the phenomena without considering any hypothetical details in the system considered. In observing the behaviour of electrons, protons and atoms we may say that what is observed

consists of spectral frequencies, intensities and states of polarization. We do not observe positions, velocities or frequencies in electronic orbits and it is in our equations connecting these unobserved quantities that difficulties arise. Heisenberg proposed to omit these details from his equations and to develop a system of calculation in which only observed quantities occur. This method calls to mind the principle of relativity in which there is no mention of the æther. The æther is a medium whose properties escape observation and in describing phenomena by the principle of relativity any mention of the æther is avoided, but just as no relativist can deny the existence of the æther so no follower of Heisenberg's principles may deny that electrons have orbital frequencies or positions.

In classical physics we have an example of a similar method of procedure in thermodynamics and it is clear from experience that such methods are very powerful, outlasting the detailed methods in which a model is assumed.

Heisenberg's method led to the application of the matrix analysis to these problems and at the same time guided by the same principles Dirac applied his notation.

It is not our purpose to give an account of matrix mechanics, but to turn our attention to the work of de Broglie and Shroedinger. This is the explanation of the phenomena by means of wave mechanics and is a return to classical methods as we shall show in the following pages. Though so different in origin, it has been shown by Shroedinger that the matrix methods follow from the wave theory of mechanics. It may be that the two views are actually one and the same, but the underlying principles are different: the one aims at describing natural phenomena by means of a new calculus which is adapted to discontinuities; while the other makes use of the extensively developed methods of the theory of vibrations and waves and applies them to the atomic problem.

CHAPTER I

THE VARIATION PRINCIPLES OF DYNAMICS AND OPTICS

WE begin our study by an examination of the variation principle of Maupertuis and Lagrange to which we have referred.

It will be necessary to go over well-known ground in order to find the starting-place for our venture upon the new theory.

It is our purpose to show that dynamics presents a close analogy to geometrical optics, but that while the latter is extended and reinforced by physical optics, the former has received no corresponding extension. This extension is now taking place under the name ' wave mechanics '.

The Principle of Least Action

A particle with velocity $\mathbf{v}$ (components v_x, v_y, v_z) has mass m, given by $m = \dfrac{m_0}{\left(1-\dfrac{v^2}{c^2}\right)}$, and momentum $m\mathbf{v}$.

The fundamental equations of Mechanics are :

$$\frac{d}{dt}(mv_x) = \mathrm{X} = -\frac{\partial \mathrm{V}}{\partial x},$$

$$\frac{d}{dt}(mv_y) = \mathrm{Y} = -\frac{\partial \mathrm{V}}{\partial y},$$

$$\frac{d}{dt}(mv_z) = \mathrm{Z} = -\frac{\partial \mathrm{V}}{\partial z},$$

where X, Y and Z are the force components, which are assumed to have a potential V.

Let us consider the variation of the integral:

$$\int_{A_1}^{A_2} (mv_x dx + mv_y dy + mv_z dz)$$

where the path of integration is represented by a path $A_1\ A_2$.

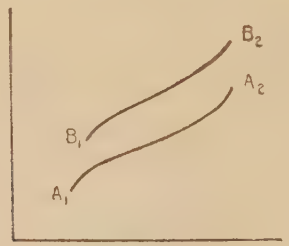

FIG. 1.

Let the components of A_1 and A_2 be $(x_1\ y_1\ z_1)$ and $(x_2\ y_2\ z_2)$ respectively. By the variation of the integral we mean the change in its value when it is taken over a neighbouring path $B_1\ B_2$ (infinitesimally close to $A_1\ A_2$) instead of over $A_1\ A_2$. This difference or variation is denoted by

$$\delta \int_{A_1}^{A_2} m(v_x dx + v_y dy + v_z dz).$$

Let the co-ordinates of B_1 be $(x_1 + \Delta x_1,\ y_1 + \Delta y_1,\ z_1 + \Delta z_1)$ and of B_2 $(x_2 + \Delta x_2,\ y_2 + \Delta y_2,\ z_2 + \Delta z_2)$.

For convenience we shall write **G** for the momentum $m\mathbf{v}$ and the components are G_x, G_y, G_z.

All the quantities G_x, G_y, G_z and x, y, z are functions of the time, t, and in the operation denoted by δ we suppose that t remains unchanged. This is explained easily by means of a diagram showing two neighbouring curves in a plane $(\xi,\ t)$.

Let the curve $a_1\ a_2$ represent $\xi = f(t)$, and let $b_1\ b_2$ represent $\xi = f(t) + g(t)$ where $g(t)$ is an infinitesimally small function of t. In considering the value $\delta\xi$ we

suppose t to remain constant, i.e. $\delta\xi = g(t) = \text{AB}$ in the diagram.

In the integral dx, dy, dz, are functions of t and will vary in the passage from one curve to the other, so that in considering the variation of the integral we shall have

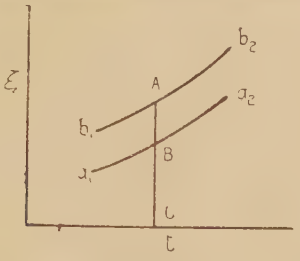

FIG. 2.

to take this into account. For the purpose of evaluating the variation we may consider $\delta \int_{A_1}^{A_2} G_x\, dx$. If we keep in mind the meaning of the integral, that it is the sum of a series, we see that this expression is the sum of terms of which $\delta(G_x dx)$ is the type. Thus the variation may be written $\Sigma\delta(G_x dx)$, the summation extending over the interval of time which elapses between A_1 and A_2.

Now $\delta(G_x dx) = \delta G_x . dx + G_x . \delta dx$.

Now when a function ξ becomes $\xi + \delta\xi$, $d\xi$ becomes $d(\xi + \delta\xi)$, so that the change in $d\xi$, which by definition is $\delta d\xi$, is equal to $d\delta\xi$.

Hence $\delta(G_x dx) = \delta G_x . dx + G_x . d\delta x$.

Thus
$$\delta \int_{A_1}^{A_2} G_x dx = \int_{A_1}^{A_2} \delta G_x dx + \int_{A_1}^{A_2} G_x . d\delta x$$
$$= \int_{A_1}^{A_2} \delta G_x . dx + [G_x . \delta x]_{A_1}^{A_2} - \int_{A_1}^{A_2} dG_x . \delta x$$
$$= [G_x \delta x]_{A_1}^{A_2} + \int_{A_1}^{A_2} (\delta G_x . dx - dG_x . \delta x).$$

For the variation of the integral including the terms in y and z, we have merely to add similar terms to this expression; these we omit for the sake of simplicity.

In the integrated portion of this expression, δx has the values in the limits corresponding to the changes from A_1 to B_1 and from A_2 to B_2 respectively. If the curve A_1A_2 represents a sequence of values which satisfy the mechanical equations, the last expression can be simplified.

In this case $v_x = \frac{dx}{dt}$, $\frac{dG_x}{dt} = -\frac{\partial V}{\partial x}$, etc., the variables x, y, z and t satisfying the curve A_1A_2 as well as these mechanical equations.

Hence $\delta G_x = \delta m.v_x + m\delta v_x.$

$$m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}; \quad v^2 = v_x^2 + v_y^2 + v_z^2.$$

$$\delta m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{v_x\delta v_x + v_y\delta v_y + v_z\delta v_z}{c^2}\right).$$

$$\delta G_x.dx = \delta G_x.\frac{dx}{dt}.dt = \delta m.v_x^2dt + mv_x\delta v_x.dt.$$

Thus summing for x, y and z

$$\Sigma\delta G_x.dx = \delta m.v^2dt + m\Sigma v_x\delta v_x.dt.$$

$$= \delta m.v^2dt + c^2\left(1 - \frac{v^2}{c^2}\right)\delta m.dt = c^2.\delta m.dt.$$

We have also

$$\Sigma dG_x.\delta x = \Sigma\frac{dG_x}{dt}.\delta x dt = \Sigma\left(-\frac{\partial V}{\partial x}.\right)\delta x.dt.$$

Hence $\Sigma(\delta G_x.dx - dG_x.\delta x) = \left(c^2\delta m + \Sigma\frac{\partial V}{\partial x}.\delta x.\right)dt$

$$= \delta(mc^2 + V).dt.$$

But $mc^2 + V = W$, the total energy of the particle, so that :

$$\delta \int_{A_1}^{A_2} (G_x dx + G_y dy + G_z dz) =$$

$$\left[G_x \delta x + G_y \delta y + G_z \delta z \right]_{A_1}^{A_2} + \int_{A_1}^{A_2} \delta W . dt \quad . \quad . \quad (1{\cdot}1)$$

This relation holds for the variation of the integral from a mechanically possible path to any neighbouring path.

If now we make A_1 and B_1 coincide and likewise A_2 and B_2, so that the variation is from one path to another possessing the same initial and final points, and if the same energy is associated with both paths so that $\delta W = 0$, then we have :

$$\delta \int_{A_1}^{A_2} (G_x dx + G_y dy + G_z dz) = 0.$$

In this case the integral has an extreme value for the mechanically possible path.

If we take the case of Newtonian mechanics where the mass is constant, $G_x dx = m v_x^2 dt$, and the integral becomes :

$$\delta \int_{A_1}^{A_2} 2T dt = 0 \quad . \quad . \quad . \quad . \quad (1{\cdot}2)$$

where T is the kinetic energy of the particle.

$\int_{A_1}^{A_2} T dt$ is spoken of as the action of the system and the expression (1·2) is the principle of least action, the oldest variation principle of dynamics. The equation (1·1) is the form in which we must leave the principle in the theory of relativity.

It is worth while to state this principle in words and to point out the very special circumstances under which it applies. The principle of least action states that the

integral $\int \Sigma G_x dx$, when taken between two points along a mechanically possible path is less than when taken between the same two points along a neighbouring path with which the same amount of energy is associated.

The proof given here shows that the integral has a stationary value, not necessarily a minimum value, and the principle should be described as a stationary principle not as a minimum principle.

The Hamilton Principle

We may write $G_x dx = G_x \cdot \frac{dx}{dt} \cdot dt$, and consequently

$$\int_{A_1}^{A_2} (G_x dx + G_y dy + G_z dz) = \int_{A_1}^{A_2} (G_x v_x + G_y v_y + G_z v_z) dt.$$

$$= \int_{A_1}^{A} \Sigma G_x v_x . dt.$$

Thus from (1)

$$\delta \int_{t_1}^{t_2} \Sigma G_x v_x . dt = \left[\Sigma G_x \delta x \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \delta W . dt \quad . \quad (1{\cdot}3)$$

where t_1 and t_2 are the times appropriate to A_1 and A_2 respectively.

But $$\delta \int_{t_1}^{t_2} W dt = W_2 \delta t_2 - W_1 \delta t_1 + \int_{t_1}^{t_2} \delta W . dt \quad . \quad (1{\cdot}4)$$

where W_1 and W_2 denote the values of W at A_1 and A_2 and δt_1 and δt_2 denote the variations in t from A_1 to B_1 and from A_2 to B_2 respectively. By the principle of energy the value of this quantity is constant for the original path, which is consistent with the mechanical equations, so that $W_1 = W_2 = W$. It should be noted that δW is the change in W in passing from this path to a neighbouring one.

From (1·3) and (1·4) we may deduce directly by writing $L = \Sigma G_x v_x - W$.

$$\delta \int_{t_1}^{t_2} L dt = \left[\Sigma G_x \delta x\right]_{t_1}^{t_2} - \left[W \delta t\right]_{t_1}^{t_2}. \quad . \quad (1\cdot5)$$

L is called the Lagrangian function and has a simple interpretation in Newtonian mechanics.

As we have seen, $\Sigma G_x v_x = 2T$ and W is the total energy $T + V$, so that $L = T - V$, the difference between the kinetic and potential energy. In the special relativity mechanics :

$$L = \Sigma m v_x^2 - mc^2 - V = m(v^2 - c^2) - V$$

$$= -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - V.$$

In the particular case when the variation in (1·5) is referred to two paths with the same initial and final points, i.e. $\delta x_1\ \delta y_1\ \delta z_1$ and $\delta x_2\ \delta y_2\ \delta z_2$ are all zero, the first term on the right of (1·5) vanishes. If in addition the time corresponding to each path is the same so that $\delta t_2 = \delta t_1$, the second term on the right vanishes and we have another integral possessing an extreme value on the mechanical path, viz. $\int_{t_1}^{t_2} L dt$.

This theorem, which is known as Hamilton's Principle, is, like the principle of Least Action, subject to close restriction. It is true for the variation from a mechanically possible path to a neighbouring one with the same end points, the time intervals associated with these paths being the same. The addition of a constant to L will not affect the vanishing of the variation under these conditions since $\delta \int_{t_1}^{t_2} (\text{const.}) dt = 0$ under the restrictions laid down.

We may thus define L as $m_0 c^2 - m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - V$ without affecting Hamilton's Principle, and we have the

advantage that for small values of v, $L = \frac{1}{2}m_0v^2 - V = T - V$, or L then becomes the Newtonian expression.

The Variation Principle of Optics

The optical principle corresponding to the mechanical principles just considered is older than these, and is the principle of Fermat. Comparison between them shows a very close analogy between these two branches of study, a point upon which Hamilton laid stress but which has been overlooked until recent times. The principle states that a ray of light in passing from a point A_1 to a point A_2 will describe a path for which the time of transit has a stationary value. If u denote the velocity of light and ds an element of the path :

$$\delta \int_{A_1}^{A_2} \frac{ds}{u} = 0.$$

This relation holds for variation from the optically possible path just as the corresponding relations in mechanics hold for variation from the mechanically possible path.

The actual value of $\int_{A_1}^{A_2} \frac{ds}{u}$ in this case depends on the co-ordinates of A_1 and A_2, and we may denote it by $V(x_1y_1z_1\ x_2y_2z_2)$.

The value is also determined by the time interval from A_1 to A_2 for

$$\int_{A_1}^{A_2} \frac{ds}{u} = \int_{A_1}^{A_2} dt = t_2 - t_1.$$

We thus have :

$$V(x_1y_1z_1\ x_2y_2z_2) = t_2 - t_1 \quad . \quad . \quad . \quad (1{\cdot}6)$$

This is strictly geometrical optics since we are considering a ray, but we can pass from it to the wave theory.

Let A_1 be a fixed point and denote $x_2y_2z_2$ by xyz. Let t_1 be put equal to zero, the times being measured from this instant and write $t_2 = t$.

The equation (1·6) may now be written :

$$V(xyz) = t \quad . \quad . \quad . \quad . \quad . \quad (1·7)$$

This gives the locus of points which represent the ends of rays starting from a source A_1 which satisfy the optical principle, and may be taken to represent a wave surface at the instant t.

Analogy between the Mechanical and Optical Principles

There must clearly be a wave theory associated with the mechanical theorems corresponding to the wave theory in optics. We have mentioned that this analogy was pointed out by Hamilton, but it has remained to de Broglie and Shroedinger to appreciate the importance of this in the realm of atomic physics. In optics we have extended our theory by means of the wave theory of light, and have built up a satisfactory theoretical basis to account for optical phenomena in the cases where the dimensions are small.

We find no difficulty in accepting as parts of a single consistent theory the subjects of geometrical optics and diffraction phenomena. We know exactly what is to be understood by the statement that light travels in straight lines and we do not feel compelled to speak of classical and non-classical optical theories when we discover a whole range of phenomena in which light bends round edges.

The mechanical wave theory opens up a new field corresponding to diffraction in optics, and we may find a solution of our difficulties in mechanical problems on a small scale by means of our optical analogy. The success already attained in this direction where our practical examples are afforded by the study of the atom has made this branch of mechanics sufficiently important to receive the name of Wave Mechanics or the New Quantum Theory.

We may show the derivation of the mechanical wave

theory by proceeding from the principle of least action in the same way as we arrive at the optical wave theory from Fermat's principle.

Denote the extreme value of the integral, $\int_{t_1}^{t_2} (G_x v_x + G_y v_y + G_z v_z) dt$, along the path of which the energy, W, is associated by the function S.

S depends on the co-ordinates of A_1 and A_2 and upon W.

Thus for the extreme value :

$$S(x_1\, y_1\, z_1,\ x_2\, y_2\, z_2,\ W) = \int_{t_1}^{t_2} \Sigma G_x v_x . dt$$

and according to (1·1)

$$\delta S = \left[\Sigma G_x \delta x\right]_{A_2} - \left[\Sigma G_x \delta x\right]_{A_1} + \int_{t_1}^{t_2} \delta W . dt \quad (1{\cdot}8)$$

In this process of passage from one path to another we have assumed that the energy, W, associated with the path is constant all along it; for one path the constant value is W, for the other it is $W + \delta W$; thus the integral in the last expression is equal to $\delta W(t_2 - t_1)$ since δW is a constant change all along the path.

Hence

$$\delta S = \left[\Sigma G_x \delta x\right]_{A_2} - \left[\Sigma G_x \delta x\right]_{A} + \delta W(t_2 - t_1) \quad (1{\cdot}9)$$

But $$\delta S = \Sigma \frac{\partial S}{\partial x_2} . \delta x_2 + \Sigma \frac{\partial S}{\partial x_1} \delta x_1 + \frac{\partial S}{\partial W} . \delta W \ . \quad (1{\cdot}10)$$

since by definition S is a function of $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$ and W.

Thus by comparison of (1·9) and (1·10),

$$\frac{\partial S}{\partial x_2} = G_{x2} \qquad \frac{\partial S}{\partial x_1} = - G_{x_1} \qquad \frac{\partial S}{\partial W} = t_2 - t_1 \quad (1{\cdot}11)$$

and there are similar relations in y and z.

Of these we require for our present purpose :

$$\frac{\partial S}{\partial W} = t_2 - t_1.$$

As in the optical case we may consider $x_1\, y_1\, z_1$ as our origin and put $t_1 = 0$ as our time reference. Then $x_2\, y_2\, z_2$ is the end of our track and lies on the locus :

$$\frac{\partial S}{\partial W} = t \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}11)^1$$

corresponding with the optical equation (1·6).

Equation (1·11) is not derived quite so simply as equation (1·6), nor has it so simple a form, but there is no difference in principle.

We have now a surface associated with our mechanical track, and the mechanical problem is analogous to the optical one.

The same result may be derived from Hamilton's Principle.

In this case we consider the integral : $\int_{t_1}^{t_2} L dt$ along a path joining A_1 and A_2 associated with a time interval $(t_2 - t_1)$ which we will denote by t.

Let the extreme value of this integral be denoted by S^*, and to show its dependence on $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$ and t we write :

$$S^*(x_1\, y_1, z_1,\ x_2\, y_2\, z_2,\ t) = \int_{t_1}^{t_2} L dt.$$

From (1·5) $\Sigma \frac{\partial S^*}{\partial x_2}\delta x_2 + \Sigma \frac{\partial S^*}{\partial x_1}\delta x_1 + \frac{\partial S^*}{\partial t}\delta t$

$$= \Sigma G_{x_2}\delta x_2 - \Sigma G_{x_1}\delta x_1 - W\delta t$$

Hence :

$$\frac{\partial S^*}{\partial x_2} = G_{x_2}, \quad \frac{\partial S^*}{\partial x_1} = -\, G_{x_1}, \quad \frac{\partial S^*}{\partial t} = -\, W \quad (1{\cdot}12)$$

and similar equations in y and z.

Again we fix our attention on one of these, viz. : $\frac{\partial S^*}{\partial t} = -\, W$, which like (1·11) gives the locus of $x_2\, y_2\, z_2$

associated with a time t measured from A_1 as origin of $(x\,y\,z)$ and $t_1 = 0$ as origin of time.

This extension of mechanics raises a number of interesting questions on the interpretation of material points and quantities associated with them, like energy and momentum. In a wave theory we deal with quantities having extension in space, and while we are familiar with particles, with energy and with momentum in ordinary mechanics we have yet to become familiar with them in the extended sense.

The view taken in wave mechanics is that the material point is a singularity associated with something which is periodic and extended in space; the singularity corresponds to the mass-point of ordinary mechanics.

In the theory of relativity we learn that a quantity of energy, W, is associated with a mass $m = \frac{W}{c^2}$, and it is according to this view unnecessary to distinguish between mass and energy. If W_0 denote the energy of a particle of mass m_0 in a system at rest with respect to the particle then we know that the energy with respect to a system, with respect to which the particle is moving with velocity v, is given by :

$$W = mc^2 = \frac{m_0 c^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{W_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}.$$

Suppose that the periodicity of the particle in the system in which it is at rest is ν_0. It follows from the theory of relativity that in a system, in which it is moving with velocity v, the phenomenon appears as a wave of frequency ν, where ν is given by the formula :

$$\nu = \frac{\nu_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

Thus W and ν change in the same way and we are led to write :

$$W \propto \nu$$

$$\text{or } W = k\nu$$

where k = constant.

Now we know that the equation $W = h\nu$ where $h = 6{\cdot}55 \times 10^{-27}$ C.G.S. units is a relation between energy and frequency which is amply justified in modern experimental physics, so that our constant k may be interpreted as Planck's universal constant h.

There is then no difficulty in understanding how we are to interpret W in the space away from the particle. We must suppose that there is some quantity characteristic of the mechanical wave which at the point which we usually regard as the position of the particle is interpreted physically as the energy of the particle, but which, in the field about this point, is interpreted physically as a frequency ; the frequency of the mechanical wave.

The momentum is again a quantity for which we have a well-known interpretation when associated with a particle. We have seen that the x-component, G_x, may be made equal to $\frac{\partial S}{\partial x}$. If the surface S is to have a significance in the field corresponding to that of a wave in optics then the interpretation of momentum in the field becomes clear, it is a vector quantity with components, $\frac{\partial S}{\partial x}$, $\frac{\partial S}{\partial y}$, $\frac{\partial S}{\partial z}$, i.e. it is directed along the normal to the surface S = constant.

Another important quantity which we may add to the above is the action. We have seen that in Newtonian mechanics action is measured by $\int \Sigma G_x v_x dt$ and the surfaces S = constant are surfaces over which this quantity is a constant. Since we know that action is to be measured in terms of a fundamental unit, h,

it is natural to imagine the space filled with surfaces of constant action arranged so that the values of this constant progress from the lowest value h through successive integral values. The constant h provides us with a natural interval for our graduated series of surfaces.

The Canonical Equations and the Hamilton-Jacobi Equation

Let us suppose that the tracks of the particle which we are considering are drawn for short intervals τ and $\tau + \delta\tau$. These times correspond to the track A_1A_2 and the neighbouring varied track B_1B_2 respectively.

We may write for the co-ordinate x_2 of A_2 the following equation :

$$x_2 = x_1 + v_x\tau$$

v_x denoting the x-component of the velocity, supposed uniform, over the small interval τ corresponding to the track A_1A_2.

For the displaced track B_1B_2 we have :

$$x_2 + \delta x_2 = x_1 + \delta x_1 + v_x\tau + \delta(v_x\tau)$$

and hence : $\delta x_2 = \delta x_1 + v_x\delta\tau + \tau\delta v_x.$

Now $S = \int_{t_1}^{t_2} \Sigma G_x v_x . dt = (\Sigma G_x v_x)\tau$ in this particular case, so that on making the variation :

$$\delta S = \Sigma G_x v_x \delta\tau + \Sigma(G_x \delta v_x + \delta G_x . v_x)\tau \quad . \quad (1{\cdot}13)$$

From equation (1·9)

$$\delta S = (\Sigma G_x \delta_x)_2 - \Sigma(G_x \delta x)_1 + \delta W . \tau$$

and $$(G_x)_2 = (G_x)_1 + (\dot{G}_x)_1\tau.$$

so that the second expression for δS may be written :

$$\delta S = (\Sigma \dot{G}_x \delta x - \Sigma G_x \delta v_x - \delta W)\tau + \Sigma G_x v_x . \delta\tau \quad (1{\cdot}14)$$

and it follows from (1·13) and (1·14) that :

$$\delta W = \Sigma v_x o G_x - \Sigma \dot{G}_x \delta x \, . \quad . \quad . \quad (1{\cdot}15)$$

From (1·15) we deduce

$$\frac{\partial W}{\partial G_x} = v_x \text{ and } \frac{\partial W}{\partial x} = -\dot{G}_x \quad . \quad . \quad (1\cdot16)$$

together with two other pairs of equations for y and z.

These six equations are the canonical equations, and in deriving them we must express W in terms of the co-ordinates and momentum.

It is customary to write $(q_1q_2q_3)$ for the co-ordinates and $(p_1p_2p_3)$ for the corresponding components of momentum, when we may write (1·16) thus :

$$\dot{q}_m = \frac{\partial H}{\partial p_m}, \; \dot{p}_m = -\frac{\partial H}{\partial q_m}. \quad . \quad . \quad (1\cdot16)^1$$

where m may have the values 1, 2, 3 and H denotes the energy expressed in terms of the co-ordinates and momentum. If, for the sake of simplicity, we limit ourselves to one variable x or q, we may express this by the equation :

$$H(q, \; p) = W.$$

The mechanical problem is thus reduced to the determination of the energy W in terms of the co-ordinates and momentum, the derivation of the canonical equations and their solution.

We may proceed one step farther, but we must content ourselves with a statement of the procedure and omit the argument.

If we denote time by t, we have from the third of the equations (1·12)

$$\frac{\partial S^*}{\partial t} + W = 0$$

and this we may now write :

$$\frac{\partial S^*}{\partial t} + H(q, \; p) = 0.$$

From the first of the equations (1·12) we have $p = \dfrac{\partial S^*}{\partial x}$ regarding x_2 as any co-ordinate on the track of the particle.

Thus S must satisfy the equation :

$$\frac{\partial S^*}{\partial t} + H\left(q, \frac{\partial S^*}{\partial q}\right) = 0 \quad . \quad . \quad . \quad (1{\cdot}17)$$

In general H depends explicitly on the time, but when this is not the case we may write :

$$S^* = -Et + F$$

where F is some function of the co-ordinates and E is written for a constant appropriate to the problem.

Thus from (1·17) we have :

$$H\left(q, \frac{\partial S^*}{\partial q}\right) = H\left(q, \frac{\partial F}{\partial q}\right) = E \quad . \quad . \quad (1{\cdot}17)^1$$

so that E is the total energy of the system and is constant.

$(1{\cdot}17)^1$ is the simpler form of the Hamilton-Jacobi equation, (1·17) being the more general form.

The equation is of fundamental importance in classical mechanics and may be used, as we shall see, in the derivation of the wave equation which is the corner stone of the new theory. The Hamilton-Jacobi equation may be regarded as an approximation to the wave equation.

The importance of this equation is that once a solution of (1·17) is determined in the form $S^*(t, q, \alpha)$, where α is a constant of integration, then the solution of the canonical equations is given by :

$$\frac{\partial S^*}{\partial \alpha} = \beta \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}18)$$

where β is another constant.

In the case we have been considering, viz. that of a particle, we have three co-ordinates and three components of momentum, and there are then three equations like (1·18) containing three constants α and three constants β.

Generalized Co-ordinates

We have used the co-ordinates (x, y, z) in the foregoing pages, and these are the simplest from the physical point of view. They correspond exactly with ordinary physical space. They are, however, not always the most convenient. We shall find them sufficient for our purpose in some cases, but since we have to deal with systems and not merely mass points we must consider briefly the more convenient methods in use for dealing with the mechanics of systems.

It is found that collections of particles or systems of mass points such as rods, rotators consisting of one particle describing a path round another, and even more complicated structures, can be described with regard to their position and motion by means of a number of independent variables, not of course necessarily x, y and z; the number and nature of these depending on the form of the structure.

The position of a single point is most simply described by stating its distance from three fixed mutually perpendicular planes, by stating in fact its x, y and z co-ordinates. In this case three co-ordinates are necessary and three are sufficient; the particle is said to have three degrees of freedom.

If we wish to fix the position of a rod, AB, we may proceed as follows. Consider a vertical line through its centre of gravity, G, and a plane, S, through this line parallel to some fixed plane. Draw a perpendicular from B on the vertical line and let the foot of the perpendicular be C. If we know the position of the centre of gravity, the inclination of the rod to the vertical and the inclination of the plane GBC to the plane, S, we know the position of the rod exactly, neglecting of course the possibility of rotation of the rod about its long axis.

We thus require five quantities to describe the position of the rod and these are both necessary and sufficient, and they are independent of one another; any

one of them may vary and the others remain unaltered. These five quantities, say, x, y, z, θ and ϕ, are called the generalized co-ordinates of the rod, and we can describe its mechanics by means of them ; the rod is also said to have five degrees of freedom.

It is customary to denote the co-ordinates by $q_1 q_2 \ldots q_n$ where n is their number. In the case of a particle each position is given by a set of values (xyz), and in the case of a system each position is given by a set $(q_1 q_2 \ldots q_n)$; the former case is represented by a point in 3-dimensions, the latter could be represented by analogy by a point in n-dimensions. We should, however, regard the hypothetical n-dimensional space as a mere convenience, we should not give it a physical significance ; the phenomenon is still one of three dimensions.

The kinetic energy of the system can be expressed in terms of the generalized velocities $\frac{dq}{dt}$ or $\dot{q}$, and in the important cases the kinetic energy, T, is a quadratic in the $\dot{q}$'s.

The momentum corresponding to a co-ordinate q_r is defined to be :

$$p_r = \frac{\partial \mathrm{T}}{\partial \dot{q}_r}$$

the differentiation being made with respect to $\dot{q}_r$, the other $\dot{q}$'s being regarded as constant in the operation.

These relations for the p's allow us to solve for the $\dot{q}$'s in terms of the p's, and then T can be expressed in terms of the p's instead of the $\dot{q}$'s so that T is a function of the p's and the q's.

This may be illustrated by the case of a particle moving in a plane about the origin, and by describing the position by polar co-ordinates r and θ these will correspond to q_1 and q_2.

The velocity components are $\dot{r}$ and $r\dot{\theta}$, and we have in this case

$$\mathrm{T} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2$$

which we may write :

$$T = \tfrac{1}{2}m\dot{q}_1^2 + \tfrac{1}{2}mq_1^2\dot{q}_2^2$$

By our definition :

$$p_1 = \frac{\partial T}{\partial \dot{q}_1} = m\dot{q}_1$$

$$p_2 = \frac{\partial T}{\partial \dot{q}_2} = mq_1^2\dot{q}_2.$$

Thus we may write :

$$T = \frac{1}{2m}p_1^2 + \frac{1}{2m}\frac{p_2^2}{q_1^2}$$

T in this case being a quadratic function of p_1 and p_2 but containing also q_1. It happens that q_2 (or θ) does not occur in this example, but in general T is a function of all the p's and q's.

There is a very remarkable extension of the theorems we have considered for the case of three co-ordinates to the case of n co-ordinates. The theorem of Maupertuis (1·2) holds for systems as for particles, in fact the case we have considered is a special case of the former.

T must be expressed in terms of the q's and $\dot{q}$'s, and the points A_1 and A_2 which represented two positions $(x_1y_1z_1)$ and $(x_2y_2z_2)$ of the particle now represent two sets of values of the generalized co-ordinates which we may write $(q_1q_2 \ldots q_n)_1$, $(q_1q_2 \ldots q_n)_2$ or we may use our geometrical analogy and say that A_1 and A_2 are two points of our n-dimensional space.

The path of integration was a three-dimensional track along which the particle moved under the mechanical conditions of the problem and we may now say in a convenient geometrical language that the path of integration is along an n-dimensional curve consistent with the mechanics of the system.

In the same way we may determine W, the total energy of the system and form the expression :

$$L = \Sigma p\dot{q} - W$$

which is analogous to our former expression L in equation (1·5).

Then Hamilton's theorem :

$$\delta \int_{t_1}^{t_2} \mathrm{L} dt = 0$$

holds for the system as for the particle.

The generalized equations of the system are given by (1·16)[1] except that m in those equations may have the values 1 to n.

Finally we have a generalized Hamilton-Jacobi equation (1·17) and a set corresponding to (1·18), in fact if we pass in imagination from 3 to n dimensions we may pass from the mechanics of a particle to the mechanics of a system.

In making the extension contemplated in wave mechanics we have to take over our geometrical ideas of three dimensions and apply them to n-dimensions. The surfaces corresponding to systems are n-dimensional, where n is equal to the number of degrees of freedom of the system. This we must regard as merely a convenient mode of expression, and when we speak of surfaces and waves in n-dimensions we speak by analogy. The view expressed in the new theory is that waves are to be associated with fundamental units of mass like protons and electrons, and that these waves have a physical existence. A system is a complicated collection of waves associated with the particles of which the system is composed, but the wave problems concerning systems cannot be solved by considering the individual waves, we have to use what we may describe as the generalized wave theory. It will be seen from this that this generalized wave theory bears to the wave theory of particles the same relation as generalized mechanics bears to the mechanics of a particle.

CHAPTER II

MECHANICAL WAVES

IN this chapter we propose to develop the theory we have just been considering in order to bring out the wave character of mechanics in detail. This is Shroedinger's theory, and we shall finally arrive at the wave equation which represents a new condition to be introduced into physics. We shall develop our theory by considering a property of the surfaces on which S and S* of the last chapter are constant.

By the definition of S and S* it follows that a relation exists between them.

For $$S = \int_{t_1}^{t_2} (\Sigma G_x v_x) dt \quad \text{and} \quad S^* = \int_{t_1}^{t_2} L dt$$

where $$L = \Sigma G_x v_x - W.$$

Thus $$S^* - S = - \int_{t_1}^{t_2} W dt = - W(t_2 - t_1)$$

since W is supposed to be constant along the path of integration. If we take the case where $t_1 = 0$ and t_2 = a time t, then

$$S^* = S - Wt \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

These quantities are very important, and it is worth while to fix our ideas by considering a particular case.

Let a particle of mass m be set in motion vertically from the origin 0 at a time $t = 0$ with velocity u. In this case the origin is the point A_1 and the point A_2 is some point on the vertical axis. Let the co-ordinate of

this point be z and let the corresponding time be t. In this case

$$S = \int_{A_1}^{A_2} (\Sigma G_x v_x) dt = \int_0^t m\dot{z}\ddot{z}dt = \int_0^z m\dot{z}dz.$$

If g is the acceleration due to gravity $\dot{z}^2 = u^2 - 2gz$ so that

$$S = \int_0^z m(u^2 - 2gz)^{\frac{1}{2}} dz = \frac{mu^3}{3g} - \frac{m}{3g}(u^2 - 2gz)^{\frac{3}{2}}$$

Since $$L = T - V = \tfrac{1}{2}m\dot{z}^2 - mgz$$

$$S^* = \int_0^t \tfrac{1}{2}m(\dot{z}^2 - 2gz)dt.$$

The expression under the sign of integration can readily be expressed in terms of t, and on integrating we find :

$$\begin{aligned} S^* &= \frac{mu^3}{3g} - \tfrac{1}{2}mu^2t - \frac{m}{3g}(u - gt)^3 \\ &= \frac{mu^3}{3g} - \frac{m}{3g}(u^2 - 2gz)^{\frac{3}{2}} - \tfrac{1}{2}mu^2t \\ &= S - Wt \end{aligned}$$

since $W = \frac{1}{2}mu^2$ in this case. This example will serve to show how S and S* can be determined in a special case and it also illustrates equation (2·1).

Now consider in a general case the surfaces S* = constant. At any time, t, we shall consider a family of such surfaces as filling our three-dimensional space.

In the diagram let us consider three such surfaces each having the property that S* is constant over it but with a different value of the constant on each.

At any surface S_1^* the value of the function S_1^* changes with the time, the change in an interval δt being $W\delta t$ by (2·1).

If we consider the surfaces of the diagram as fixed a particular value S_1^* at time t will at time $t + \delta t$ be appropriate to some other surface in the field.

We could draw the surfaces in such a way that a particular value, say S*, would be appropriate to the first at the instant t, to the second at the instant $t + \delta t$, to the third at $t + 2\delta t$ and so on.

It would then appear as if the value S* travelled from one to the other. Since we could make δt as small as we please we could also describe this by saying that a surface S* travelled outward keeping its value constant as it progresses.

We are familiar with this in the spreading of waves where we have a surface, the wave front, progressing with the phase constant over it. Let the point A on the diagram have co-ordinates (xyz) and let B be a neighbouring point on the normal to the surface through

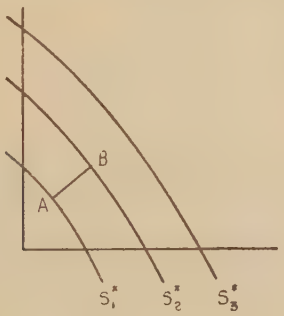

FIG. 3.

A with co-ordinates $(x + \delta x,\ y + \delta y,\ z + \delta z)$. If S* is the value of the function at A the value at B is

$$S^* + \frac{\partial S^*}{\partial x}\delta x + \frac{\partial S^*}{\partial y}\delta y + \frac{\partial S^*}{\partial z}\delta z$$ where $\frac{\partial S^*}{\partial x}$, etc., are components of the rate of change of S* along the normal, these three components combining to give the vector known as the gradient of S* and written : grad S*.

If we write δn for the element along the normal the

change in S* in passing from A to B may be written $\frac{dS^*}{dn}\delta n$, and the length of the vector, grad S*, may be written $\frac{dS^*}{dn}$ or alternatively

$$\left[\left(\frac{\partial S^*}{\partial x}\right)^2+\left(\frac{\partial S^*}{\partial y}\right)^2+\left(\frac{\partial S^*}{\partial z}\right)^2\right]^{\frac{1}{2}}$$

But it will be remembered that $\frac{\partial S^*}{\partial x}=G_x$, the component of momentum, and we have, therefore, the relation :

$$\frac{dS^*}{dn}=G \quad . \quad . \quad . \quad . \quad . \quad (2\cdot2)$$

where G is the magnitude of the total momentum, i.e. $(G_x^2+G_y^2+G_z^2)^{\frac{1}{2}}$.

We may write

$$\delta n=\frac{\delta S^*}{G}.$$

Now suppose that at A at time t the value for the surface is S*, then in order that it may appear that S* arrives at B with the same surface value at time $t+\delta t$ the condition is that the value appropriate to B is S* at this time. Thus the value at B at time t must be $S^*+W\delta t$ for then at time $t+\delta t$ it becomes $S^*-W\delta t+W\delta t$ or S*. (2·1)

Now we have seen that at a fixed time t when the value at A is S* it is $S^*+\frac{dS^*}{dn}.\delta n$ at B and so our condition is that :

$$S^*+\frac{dS^*}{dn}.\delta n=S^*+W\delta t$$

$$\text{i.e. } \frac{\delta n}{\delta t}=\frac{W}{G}.$$

Now $\frac{\delta n}{\delta t}$ represents the velocity with which the waves

travel outward along the normal to their front. This is the velocity which in optics is known as the phase velocity and since we have seen that the difference in the mechanical wave is merely that S* remains constant instead of the phase we shall describe this velocity as the phase velocity of the mechanical waves and write :

$$U = \frac{W}{G} \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}3)$$

We have now to take into account the possibility that these mechanical waves have an actual existence and if we admit this we have a field open to us in a new branch of mechanics which we know how to approach from our corresponding experience in optics.

Consideration of a Simple Form of Mechanical Wave

The most familiar form of wave in physics is :

$$y = a \sin 2\pi(\nu t - \frac{x}{\lambda} + \alpha) \quad . \quad . \quad . \quad (2{\cdot}4)$$

where a is a constant, ν the frequency, λ the wave-length and $2\pi\alpha$ the epoch, i.e. the phase at the instant $t = 0$, at the point $x = 0$. The expression $2\pi(\nu t - \frac{x}{\lambda} + \alpha)$ is called the phase.

There is a similarity between this and the expression for S* which is the phase in our mechanical waves, for it consists of a function of the co-ordinates, $-\frac{2\pi}{\lambda}x$, and a term containing t, $2\pi\nu t$. These correspond to S and $-Wt$ in the expression for S*, viz.

$$S^* = S - W(t - t_0)$$

while α corresponds to Wt_0.

The simplest form of wave we can obtain by analogy is thus :

$$y = \mathrm{A} \sin k\mathrm{S}^*.$$

Where k is a constant and has the dimensions which make $k\mathrm{S}^*$ a quantity without dimensions like the phase in the above wave.

S* is of the dimensions of action and we may thus write $k = \frac{2\pi}{h}$ where h is a constant of the dimensions of action.

Thus the simple sine form of wave may be written :

$$y = \mathrm{A} \sin \frac{2\pi}{h}(\mathrm{S} - \mathrm{W}t + \mathrm{W}t_0) \quad . \quad . \quad (2{\cdot}5)$$

By its definition S is a function of the co-ordinates and W.

By comparison with (2.4) it is evident that the frequency of this wave is :

$$\nu = \frac{\mathrm{W}}{h} . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}6)$$

Thus the mechanical quantity, W, which we interpret as the energy in dynamics, is to be interpreted as $h\nu$ in wave mechanics.

The equation (2·6) has a well-known physical significance in the quantum theory of radiation.

We have seen that the wave theory requires us to recognize that the quantities like action, energy and momentum have a significance at points away from the particle. Equation (2·6) shows that the interpretation of energy in the field is to be $h\times$ frequency of mechanical wave.

Wave Groups

Let two simple waves of the same amplitude, a, but with slightly differing frequencies ν and ν' $(= \nu + d\nu)$ and velocities U and U′ $(= \mathrm{U} + d\mathrm{U})$ travel simultaneously through a medium. They superimpose and produce a disturbance given by ;

$$y = a\left[\sin 2\pi\left(\nu' t - \frac{\nu'}{U'}x\right) + \sin 2\pi\left(\nu t - \frac{\nu}{U}x\right)\right].$$

$$= 2a \sin 2\pi\left[\frac{\nu' + \nu}{2}t - \frac{\nu'}{2}\left(\frac{1}{U'} + \frac{1}{U}\right)x\right]$$

$$\cos 2\pi\left[\frac{\nu' - \nu}{2}t - \frac{x}{2}\left(\frac{\nu'}{U'} - \frac{\nu}{U}\right)\right].$$

But $\nu' + \nu = 2\nu + d\nu$ and, neglecting $d\nu$ in comparison with ν, we write $\nu' + \nu = 2\nu$. In the same way $\nu'\left(\frac{1}{U'} + \frac{1}{U}\right)$ may be written $2\frac{\nu}{U}$. Under the cosine the quantities are of the order $d\nu$ so that first differentials must be retained.

Thus $y = 2a \sin 2\pi\left(\nu t - \frac{\nu}{U}x\right)\cos \pi\left[d\nu t - d\left(\frac{\nu}{U}\right)x\right]$.

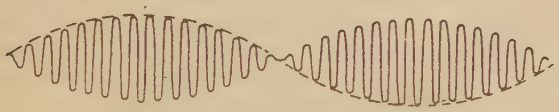

FIG. 4.

We may regard y as a wave of frequency ν and of amplitude $a \cos \pi\left\{ d\nu t - d\left(\frac{\nu}{U}\right)x \right\}$.

The amplitude varies very slowly in comparison with the frequency ν of the oscillations. The dotted curve of the figure represents the cosine term, while the full line represents the wave vibrations.

If we have a collection of waves with frequencies lying over the small interval ν to $\nu + d\nu$ we represent the vibration, y, by $a.d\nu.$ $\sin 2\pi\left(\nu t - \frac{\nu}{U}x\right)$ where a is the average amplitude for this range of frequencies.

For any finite group of waves we have :

$$y = \int_{\nu_1}^{\nu_2} a \sin 2\pi\left(\nu t - \frac{\nu}{\mathrm{U}}x\right)d\nu, \qquad . \quad (2{\cdot}7)$$

and a will in general depend on the frequency.

The dotted curve in the diagram travels with a velocity given by

$$v = \frac{d\nu}{d\left(\frac{\nu}{\mathrm{U}}\right)} = \mathrm{U} - \lambda\frac{d\mathrm{U}}{d\lambda} \quad . \quad . \quad . \quad (2{\cdot}8)$$

This can be understood by examining the cosine term which represents a wave travelling with the velocity v.

λ is the wave length corresponding to the frequency ν and v is called the group velocity.

We may apply a similar discussion to the mechanical waves. In this case the range $d\nu$ is equal to $\frac{d\mathrm{W}}{h}$ by (2·6).

For any two points A_1 and A_2 there is a series of mechanical paths corresponding to the series of values of W lying in the range W to W + dW. A group of waves corresponds to this series which, by comparing with (2·7), may be seen to be represented by :

$$y = \int_{\mathrm{W}_1}^{\mathrm{W}_2} a \sin \frac{2\pi\mathrm{S}^*}{h}d\mathrm{W} \quad . \quad . \quad . \quad (2{\cdot}9)$$

Where a, which in the optical case depends on the frequency, now depends on W and through this on the frequency of the mechanical wave. The group velocity in this case is :

$$v = \frac{d\nu}{d\left(\frac{\nu}{\mathrm{U}}\right)} = \frac{d\mathrm{W}}{d\left(\frac{\mathrm{W}}{\mathrm{U}}\right)} = \frac{d\mathrm{W}}{d\mathrm{G}}(\text{by } 2{\cdot}3) \quad (2{\cdot}10)$$

Now we have seen that $\mathrm{W} = mc^2 + \mathrm{V}$, $m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$ and it follows that $\mathrm{W} = m_0c^2\left(1 + \frac{\mathrm{G}^2}{m_0{}^2c^2}\right)^{\frac{1}{2}}$

+ V, hence remembering that the change in W which we contemplate is to be made from the first path to the adjacent second path without changing the co-ordinates, of which V is a function, we find :

$$\frac{dW}{dG} = v$$

where v is the particle velocity. Thus the group velocity is the same as the velocity of the particle, which we anticipated in (2·10) by denoting the group velocity by v.

The phase velocity, $\frac{W}{G}$, is of course different from this and we may consider the relation between them most simply by taking the case where there is no field of force.

In this case $W = mc^2$ and $G = mv$, so that :

$$U = \frac{W}{G} = \frac{c^2}{v}$$

or $Uv = c^2$. . . . . (2·11)

Since v cannot be greater than c, U cannot be less. We therefore know that energy cannot be transported by the phase waves since according to the principle of relativity energy cannot be transported with a velocity greater than that of light. The phase wave is, however, no more than a mathematical conception, the physicist contemplates groups of waves whenever energy is transported by wave motion so that energy travels with the group velocity. The particle is thus to be regarded as a group of waves and the particle velocity is in the wave theory to be interpreted as the velocity of this group.

The above consideration includes the classical case, which may be more simply considered directly, by writing $G^2 = m^2 v^2 = 2m$ T where T is the kinetic energy. We then have $G^2 = 2m(W - V)$ and from (2·10) we find at once the value of the group velocity.

The Mechanical Wave-length and Atomic Dimensions

The introduction of h in equation (2·6) was necessary from a consideration of the dimensions of the mechanical phase. There is nothing in the theory to indicate its magnitude and this could not be expected. The magnitude can only be obtained by experiment. We know that an equation of this kind has been widely applied and verified in spectroscopy and we are thus led to recognize that this constant is Planck's constant of action ($= 6{\cdot}55 \times 10^{-27}$ ergs-sec.).

In order to give a definite meaning to the frequency we must avoid giving to W a relative value as in classical mechanics, it has the value $mc^2 + V$.

Since $W = h\nu$ we have:

$$U = \frac{h\nu}{\sqrt{2m(W - V)}}$$

and

$$\lambda = \frac{U}{\nu} = \frac{h}{\sqrt{2m(W - V)}}.$$

Consider the case of an electron moving round a nucleus, as in the hydrogen atom.

Here m = mass of the electron.

$W - V$ = the kinetic energy $= \frac{1}{2}mv^2$ were v is the orbital velocity.

Thus

$$\lambda = \frac{h}{mv}.$$

The moment of momentum is mva and has the value $\frac{h}{2\pi}$ for the inner orbit according to Bohr's theory.

Thus

$$a = \frac{h}{2\pi mv}.$$

It is remarkable that the wave length is of the same order as the atomic dimensions and we may expect from our analogy that just as the theory of diffraction comes to the aid of geometrical optics when the dimen-

sions of the objects considered are of the same order as the wave length of light, so the new mechanics must be invoked to aid classical mechanics when the dimensions are of the order of the mechanical wave length which, as we have just seen, is of atomic dimensions. It is significant that our difficulties in the quantum theory have arisen in trying to apply classical mechanics to atoms.

Diffraction of the Mechanical Waves

We pass on to consider the group of waves (2·9) writing :

$$\phi = \mathrm{S}^* \quad = \mathrm{S} - \mathrm{W}t + \mathrm{W}t_0$$

and making the assumption that the interval over which W extends, i.e. the interval W_1 to W_2, is small.

For any particular mechanical track, which we may by analogy describe as a mechanical ray, there is a particular value of W, and consequently a particular value of ϕ for the mechanical wave.

Thus as we pass from wave to wave of the group ϕ will vary with W, and consequently :

$$d\phi = \frac{\partial \phi}{\partial \mathrm{W}} . d\mathrm{W}. \quad . \quad . \quad . \quad . \quad (2{\cdot}12)$$

Thus expressing the integral in terms of ϕ :

$$y = \int_{\phi_1}^{\phi_2} \frac{a \sin \frac{2\pi}{h}\phi}{\frac{\partial \phi}{\partial \mathrm{W}}} . d\phi.$$

Where a also depends on W, just as in the optical case it depends on the frequency.

If the range of integration is small we can write :

$$y = \frac{\bar{a}}{\frac{\partial \phi}{\partial \mathrm{W}}} \int_{\phi_1}^{\phi_2} \sin \frac{2\pi}{h}\phi . d\phi$$

where the bars denote average values over the range.

The value of the integral is :

$$\frac{h}{2\pi}\left(\cos\frac{2\pi}{h}\phi_1 - \cos\frac{2\pi}{h}\phi_2\right) = \frac{h}{\pi}\sin\frac{\pi}{h}(\phi_2 - \phi_1)\sin\frac{\pi}{h}(\phi_2 + \phi_1).$$

We take the range to be small enough to write $\frac{\phi_2 + \phi_1}{2} = \phi$ (an average for the extent), while $\phi_2 - \phi_1 = \Delta\phi$, the range.

Thus
$$y = \frac{\bar{a}}{\overline{\frac{\partial\phi}{\partial W}}} \cdot \frac{\sin\left(\frac{\pi}{h}\Delta\phi\right)}{\frac{\pi}{h}} \cdot \sin\frac{2\pi}{h}\phi \quad . \quad . \quad (2{\cdot}13)$$
$$= A \sin\frac{2\pi\phi}{h}$$

where A is written for the coefficient of $\sin\frac{2\pi}{h}\phi$.

The group of waves is thus compounded into a wave of phase $\frac{2\pi\phi}{h}$ with a varying amplitude. We have in fact a carrier wave $\sin\frac{2\pi}{h}\phi$ which is modulated.

Expressing $\Delta\phi$ in terms of ΔW by (2·12) we find :

$$A = \bar{a} \cdot \frac{\sin\left(\frac{\pi}{h}\overline{\frac{\partial\phi}{\partial W}}\Delta W\right)}{\frac{\pi}{h}\overline{\frac{\partial\phi}{\partial W}}}.$$

If $\overline{\frac{\partial\phi}{\partial W}} = 0$, A has a maximum value $\bar{a}\Delta W$.

This may be verified by writing $\frac{\pi}{h}\overline{\frac{\partial\phi}{\partial W}} = x$ and con-

sidering the maximum and minimum values of

$$A = \bar{a}\,\frac{\sin \alpha x}{x} \quad . \quad . \quad . \quad . \quad (2{\cdot}14)$$

where a is written for ΔW.

This expression occurs in the theory of diffraction in the calculation of the effect of superposing a series of simple harmonic vibrations of equal amplitudes and periods with phases increasing in arithmetical progression. The integration method employed here is shorter.

It follows from (2·14) that a series of maxima occur at points for which the value of x is given by

$$\tan \alpha x = \alpha x.$$

The solution of this may be obtained graphically by noting the intersections of the curve, $y = \tan \alpha x$, and the line $y = \alpha x$.

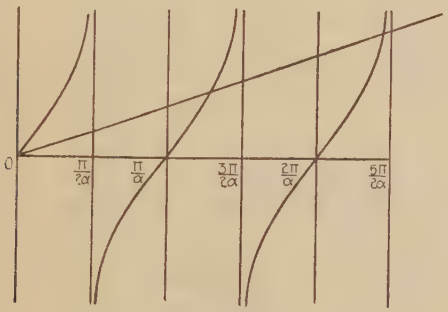

Fig. 5.

The actual values of x for these intersections are not necessary for our purpose but they are given in works on Optics.[1] From the graph it is clear that the values of αx tend to $(n + \frac{1}{2})\pi$ as n gets large.

[1] See, for example, *A Treatise on Light.* R. A. Houstoun. 5th Edn., p. 173.

If the values of A are plotted the graph obtained is as illustrated in the diagram

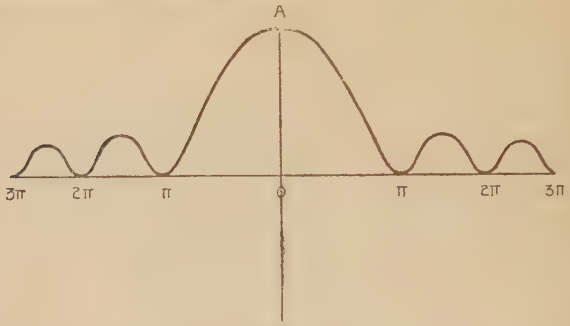

Fig. 6.

We have, therefore, in addition to the maximum $\bar{a}\Delta W$, corresponding to $\frac{\overline{\partial\phi}}{\partial W} = 0$, a series of diminishing maxima at points corresponding approximately to $\frac{\overline{\partial\phi}}{\partial W}.\Delta W = (n + \frac{1}{2})h$ with a series of zero values at points where $\frac{\overline{\partial\phi}}{\partial W}.\Delta W = nh$, where $n = 1, 2, 3$, etc.

The wave will appear approximately as represented in diagram.

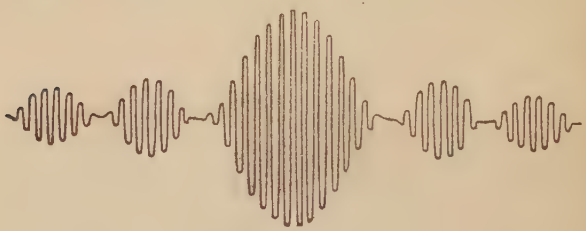

Fig. 7.

The equation $\varphi = \text{const.}$ represents the phase wave

and in the same way $\overline{\frac{\partial\varphi}{\partial W}} = \text{const.}$ represents the onward motion of the maximum values we have been considering. This moves forward with a velocity known as the group velocity.

The particular equation $\overline{\frac{\partial\varphi}{\partial W}} = 0$, the constant having in this case the particular value zero, represents the onward motion of the greatest of the series of turning values, i.e. the motion of the energy maximum.

The equation $\frac{\partial\varphi}{\partial W} = 0$ may also be written thus :

$$\frac{\partial S}{\partial W} = t - t_0$$

and is thus the same as the third of the group of equations (1·11), t being the time corresponding to the end of the track and t_0 that for the beginning. These two equations, which are thus actually the same, represent the motion of a particle, the former describing it as an energy node of a group of waves and the latter describing it as the motion of a massive particle by the classical method.

The Interpretation of the Quantum Conditions by Means of the Wave Theory

The fundamental quantum conditions $W = h\nu$ and $\oint p dq = nh$ have a simple interpretation in the wave theory.

This has been explained in the case of $W = h\nu$ by regarding W and $h\nu$ as the physical interpretation of a quantity which must be recognized throughout space, but which is recognized practically either as the energy of the particle or through the frequency of the associated wave.

The integrals in the second condition are to be taken over a complete circuit of values of q, p being the corresponding momentum.

Let us now interpret p by means of the phase wave, dq being along the line of propagation.

We shall take the case where the variables are separable, i.e. the function S consists of a sum of a number of functions each of which contains one variable only.

In the case where these are x, y, z we have :

$$S = S_1(x) + S_2(y) + S_3(z)$$

where $S_1(x)$ depends on x only, $S_2(y)$ on y only and $S_3(z)$ on z only.

If we apply the integral theorem to the x-co-ordinate we have :

$$p = G_x \text{ and } q = x,$$

hence $$\oint G_x dx = nh.$$

Suppose that a change occurs in x, y and z remaining constant, and let x undergo the small variation from x to $x + \delta x$.

We then have a surface $S^* = \text{const.}$ or

$$S - Wt = \text{const.}$$

Hence for the variation we are considering,

$$\frac{\partial S_1}{\partial x}\delta x - W\delta t - 0 \quad . \quad . \quad . \quad (2\cdot15$$

since S_1 is the only part of S which contains x.

Thus $G_x = \dfrac{\partial S_1}{\partial x} = \dfrac{W}{U_x}$ where U_x denotes the velocity of the phase wave in the x-direction.

The quantum condition thus becomes :

$$\oint \frac{W}{U_x} dx = nh$$

$$\text{or } \oint \frac{\nu}{U_x} dx = \oint \frac{dx}{\lambda_x} = n$$

where λ_x is the distance between places of equal phas along the x-direction. The integral gives the numbe of these wave lengths in the path and the quantum condition is such that a whole number of waves exis

n the permissible quantum path. They are thus in he condition to persist and are not annulled by inter- erence.

The Wave Equation

We have seen that the equation $S^* = \text{const.}$, can e regarded as representing a spreading wave surface, nd we now pass on to consider the wave equation in differential form.

In the wave theory of light we are familiar with the quation of the wave represented by (2·4). If U denote he phase velocity the equation is :

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{U^2}\frac{\partial^2\psi}{\partial t^2}, \quad . \quad . \quad . \quad . \quad (2{\cdot}16)$$

here ψ has been written in the place of y and (2·4) a solution of (2·16).

The general wave equation in three dimensions is :

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{U^2}\frac{\partial^2\psi}{\partial t^2} \quad . \quad . \quad (2{\cdot}17)$$

nd the solution of this is :

$$\psi = a \sin 2\pi\left(\nu t - \frac{lx + my + nz}{\lambda} + \alpha\right), \quad (2{\cdot}18)$$

here $(l\ m\ n)$ are the direction cosines of the normal o the plane wave (2·18). The expression on the left f (2·17) is denoted by $\nabla^2\psi$ by English writers and by $\Delta\psi$ by writers on the continent.

The equation is often expressed in another notation o understand which we must refer to the meanings of he terms divergence and gradient. The divergence f a vector **A** which has components $(A_x\ A_y\ A_z)$ is a calar quantity defined thus :

$$\text{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

The gradient of a scalar quantity, ψ, is a vector with

components, $\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\delta z}\right)$, thus the left side of (2·17) may also be written div(gradψ) and the wave equation (2·17) is written alternatively :

$$\nabla^2\psi = \triangle\psi = \frac{1}{U^2}\frac{\partial^2\psi}{\partial t^2}$$

or

$$\text{div}(\text{grad}\psi) = \frac{1}{U^2}\frac{\partial^2\psi}{\partial t^2} \quad . \quad . \quad . \quad (2\cdot19)$$

The form (2·19) is particularly useful because we can use it very conveniently in generalizing the wave equation for the purpose of applying it in the general theory of relativity, and in the case of mechanical systems. We must apply a similar equation to our mechanical waves, and all that is necessary is to introduce the appropriate phase velocity.

The mechanical wave equation is thus :

$$\text{div}(\text{grad}\psi) = \frac{G^2}{W^2}\frac{\partial^2\psi}{\partial t^2} \quad . \quad . \quad . \quad (2\cdot20)$$

In the case of waves already familiar to the physicist ψ denotes some condition of the medium such as displacement of its particles from a mean position or such as pressure, but the meaning of ψ in wave mechanics is not so easily interpreted. We shall later refer to the suggestions made with regard to the physical meaning of ψ.

The equation of a simple sine form of mechanical wave is

$$\psi = A \sin\frac{2\pi}{h}(S - Wt + Wt_0) \quad . \quad . \quad (2\cdot21)$$

in which we have replaced the familiar phase by S* or $S - Wt + Wt_0$, which we have seen is the mechanical analogy to the phase.

In this case we have :

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{4\pi^2}{h^2}W^2\psi$$

so that (2·20) is reduced to :

$$\text{div}(\text{grad}\psi) + \frac{4\pi^2}{h^2}G^2\psi = 0 \quad . \quad . \quad (2{\cdot}22)$$

From a calculation outlined earlier in this chapter (page 34) we see that

$$G^2 = m_0{}^2c^2\left\{\left(\frac{W - V}{m_0c^2}\right)^2 - 1\right\} \quad . \quad (2{\cdot}23)$$

and we must examine the form taken by (2·22) when we wish to deal only with the equation offered by classical mechanics.

In this case the kinetic energy is $\frac{1}{2}m_0v^2$, where m_0 denotes the Newtonian mass, and we speak of the energy, E, meaning by that $\frac{1}{2}m_0v^2 + V$.

E is, however, an approximation to the relativity value :

$$(m - m_0)c^2 + V$$

and this differs from the classical value by a quantity of the order $\left(\frac{v}{c}\right)^2$ which is very small in all classical problems.

We thus write

$$E = (m - m_0)c^2 + V$$

and $$W = mc^2 + V = E + m_0c^2 + V.$$

The value of G^2 may now be written :

$$G^2 = \frac{(E - V)^2}{c^2} + 2m_0(E - V) \quad . \quad (2{\cdot}24)$$

Thus in any example contemplated in the classical theory we need retain only the second term on the right of (2·24), and in the second term E may be given the usual value $(\frac{1}{2}m_0v^2 + V)$ without the introduction of any serious inaccuracy.

We may derive this more directly by writing $G^2 = m_0{}^2v^2 = 2m_0T = 2m_0(E - V)$.

The classical wave equation is thus :

$$\text{div}(\text{grad}\psi) + \frac{8\pi^2}{h^2} m_0(\text{E} - \text{V})\psi = 0 \quad . \quad (2{\cdot}25)$$

A simple case where we have only one variable, x, is :

$$\frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2}{h^2} m_0(\text{E} - \text{V})\psi = 0$$

and by means of this equation we shall illustrate a very useful rule for obtaining the wave equation.

We write down the energy equation for this problem

$$\tfrac{1}{2}m_0\dot{x}^2 + \text{V} = \text{E}$$

and express this in terms of the momentum $p = m_0\dot{x}$

$$\frac{1}{2m_0} p^2 + \text{V} = \text{E}.$$

The Hamilton-Jacobi equation is :

$$\frac{1}{2m_0}\left(\frac{\partial \text{F}}{\partial x}\right)^2 + \text{V} = \text{E} \text{ (compare } 1{\cdot}17^1$$

We may derive the wave-equation by replacing p in the energy equation or $\frac{\partial \text{F}}{\partial x}$ in the H.-J. equation by the operator $\frac{h}{2\pi i}\frac{\partial}{\partial x}$, where $i = \sqrt{-1}$.

We then obtain an operator :

$$\frac{1}{2m_0}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x}\right)^2 + \text{V}$$

which is taken to be equal to the operator E, and so by allowing the two to operate on a function ψ we have :

$$\left\{\frac{1}{2m_0}\left(\frac{h}{2\pi i}\cdot\frac{\partial}{\partial x}\right)^2 + \text{V}\right\}\psi = \text{E}\psi$$

or

$$\frac{h^2}{8\pi^2 m_0}\frac{\partial^2\psi}{\partial x^2} + (\text{E} - \text{V})\psi = 0$$

i.e.

$$\frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2 m_0}{h^2}(\text{E} - \text{V})\psi = 0.$$

This will appear a very artificial way of arriving at the wave equation, but it will be convenient to keep it in mind.

If more than one variable be present each component of momentum, p_1, p_2, p_3, must be replaced by the corresponding operators $\frac{h}{2\pi i}\frac{\partial}{\partial x}$, $\frac{h}{2\pi i}\frac{\partial}{\partial y}$, $\frac{h}{2\pi i}\frac{\partial}{\partial z}$ and the operation upon ψ must be performed.

The introduction of the wave equation is a new departure in mechanics, the conditions involved in it give additional limitations which were not foreseen in the older classical theory. The first concern is not the meaning of the quantity ψ, but rather the limitations imposed upon E by the equation and certain conditions regarding its solution. With regard to the older quantum theory it used to be said that the quantum conditions represented certain extra limitations not included in the classical theory which were like boundary conditions; this point is very clearly brought to light in the new theory.

It will be seen that the methods employed do no violence to classical dogma, they add to it, but the problem is attacked on strictly classical lines and it must be confessed that this first round in the contest has ended in a victory for classical methods. This point we hope to appreciate by considering the problems in more detail.

CHAPTER III

THE NATURE OF THE PROBLEM AND ITS SOLUTION

THE nature of the new problem which mechanics in this form brings before us is best understood by referring to cases of a similar character, of which other problems in physics afford numerous examples.

We will take the problem of the vibrating string, which may present itself in this way. At a given instant which we choose as the origin of our measurement of time the string is at rest in some stated form. Let us suppose it is stretched between two points at distance l apart and let the displacement, y, of the string be referred to the line joining these points which will be taken as the x-axis.

The string, which is under tension, is then released from its original position and allowed to vibrate, and we require to find the displacement at any point, distant x from one of the fixed points, at a given time t.

The wave motion of the string is represented by the equation :

$$\frac{\partial^2 y}{\partial t^2} = \mathrm{U}^2 \frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad (3{\cdot}1)$$

This is the wave equation in this case, and the possible solutions are infinite in number, for we know that the function :

$$y = \mathrm{A}e^{ax + bt}$$

is a solution and there is no restriction on A and only one relation between a and b, viz.: $b^2 = \mathrm{U}^2 a^2$.

Nevertheless we can obtain a definite solution because of other conditions of the problem. These are the boundary conditions.

At all values of t the displacements at $x = 0$ and at $x = l$ are zero, so also are the velocities $\frac{dy}{dt}$ at these points; and at $t = 0$ we know the value of y for all points on the string since its original deformation is given.

So far we have a wave equation in mechanics corresponding to (3·1), but we have no other conditions. These conditions are that the values of ψ must be everywhere finite, single valued and continuous, and they must approach the value zero at infinity.

The values of ψ which satisfy these conditions are described as the characteristic functions, and we shall see that only certain values may be assumed by E corresponding to these functions; these values are described as the characteristic values. Thus the problem is to find solutions of the wave equation which satisfy these conditions.

The solutions, and especially the values to which E is limited by these conditions, are of the greatest interest, and it will be worth while to consider how they are obtained in the most important examples.

It will be seen that there is no especial mathematical difficulty in deriving them, and we shall on this account begin with a discussion of the type of equation which generally occurs in atomic problems and obtain its solution.

The wave equation in these examples presents itself ultimately in the form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad . \quad . \quad . \quad (3·2)$$

where P and Q are polynomials in x.

This may be transformed into an equation which may be solved by means of a series, in fact we shall see that the equation may be put into the form:

$$(a_2x^{k+2} + b_2x^{l+2})\frac{d^2V}{dx^2} + (a_1x^{k+1} + b_1x^{l+1})\frac{dV}{dx} + (a_0x^k + b_0x^l)V = 0 \quad . \quad . \quad . \quad . \quad (3{\cdot}3)$$

where k and l are whole numbers, the particular feature of this equation being that the difference between the powers of x is the same in each bracket.

The second of these equations may be written :

$$x^k\left(a_2x^2\frac{d^2V}{dx^2} + a_1x\frac{dV}{dx} + a_0V\right) + x^l\left(b_2x^2\frac{d^2V}{dx^2} + b_1x\frac{dV}{dx} + b_0V\right) = 0 \quad . \quad (3{\cdot}4)$$

By regarding $x\frac{d}{dx}$ as an operator we can express (3·4) still more conveniently if we examine the double application of this operator, which we write $\left(x\frac{d}{dx}\right)^2$. Suppose the operation is made upon a function F, then the double application on F is written $\left(x\frac{d}{dx}\right)^2$F and this means $\left(x\frac{d}{dx}\right)\left(x\frac{dF}{dx}\right) = x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx}$.

or $$x^2\frac{d^2F}{dx^2} = \left(x\frac{d}{dx}\right)^2F - \left(x\frac{d}{dx}\right)F.$$

Thus the first bracket in (3·4) may be written :

$$\left\{a_2\left(x\frac{d}{dx}\right)^2 + (a_1 - a_2)x\frac{d}{dx} + a_0\right\}V.$$

We may define a function

$$f(z) = a_2z^2 + (a_1 - a_2)z + a_0$$

and also $$g(z) = b_2z^2 + (b_1 - b_2)z + b_0$$

and write (3·4) in the form :

$$x^kf\left(x\frac{d}{dx}\right)V + x^lg\left(x\frac{d}{dx}\right)V = 0 \quad . \quad . \quad (3{\cdot}5)$$

Divide throughout by x^k, it being supposed that $k > l$, and we obtain:

$$f\left(x\frac{d}{dx}\right)\mathrm{V} + \frac{1}{x^c}g\left(x\frac{d}{dx}\right)\mathrm{V} = 0 \quad . \quad . \quad (3{\cdot}6)$$

where c is a positive integer.

If $l > k$ we divide by x^l and obtain an equation of the same form.

The solution of (3·6) may be solved immediately by the method of integration by series (Forsyth's *Diff. Eqns.*, ch. v), and we reproduce here as much as is necessary for our purpose in order to develop completely some of the most interesting results of the theory.

Let it be assumed that:

$$\mathrm{V} = \mathrm{A}_1x^{m_1} + \mathrm{A}_2x^{m_2} + \mathrm{A}_3x^{m_3} + \text{etc.},$$

where the m's are integral numbers arranged in ascending order and the A's are constants between which certain relations exist.

The method of solution consists in substituting the value of V in (3·6) and equating to zero the coefficients of the powers of x. In this way the m's and the relations between the A's are determined, and we are left finally with a solution containing the proper number of arbitrary constants, which in our case is two.

The substitution of V will give rise to terms of the form $\left(x\frac{d}{dx}\right)^2 x^n$, where n is an integer, and this is equal to n^2x^n so that after the operation we find that $x\frac{d}{dx}$ is substituted by the index of the power of x.

Thus $f\left(x\frac{d}{dx}\right)x^n = f(n)x^n$ and on substituting the series in (3·6) we find that:

$$\mathrm{A}_1f(m_1)x^{m_1} + \mathrm{A}_2f(m_2)x^{m_2} + \ldots$$
$$+ \mathrm{A}_1g(m_1)x^{m_1-c} + \mathrm{A}_2g(m_2)x^{m_2-c} + \ldots = 0 \quad (3{\cdot}7)$$

The lowest power of x in this equation is $(m_1 - c)$,

and since the left-hand side must be identically zero the coefficient of x^{m_1-c} must vanish.

Hence $g(m_1) = 0$, for A_1 is not zero since it is assumed to be the coefficient of a term present in the series.

The roots of this equation give the values of m_1 with which the series begins, and in the case we are considering $g(z)$ is a quadratic and the number of roots is two.

On examining the other terms of the series we find that a solution can be obtained in this way by equating the indices thus :

$$m_1 = m_2 - c$$
$$m_2 = m_3 - c, \text{ etc.}$$

and this is possible since they are arranged in ascending order. From this it follows that the series advances in powers of x differing by c.

Again since the terms vanish identically we must have

$$A_1 f(m_1) + A_2 g(m_2) = 0$$
$$A_2 f(m_2) + A_3 g(m_3) = 0, \text{ etc.}$$

from which all the coefficients may be obtained in terms of the first.

Suppose that a is one of the values of m_1 with which the series begins, then we have :

$$A_2 = -\frac{f(a)}{g(a+c)}A_1, \quad A_3 = \frac{f(a+c)\,f(a)}{g(a+2c)\,g(a+c)}A_1, \text{ etc.}$$

and the solution is :

$$V = A_1 x^a \left\{ 1 - \frac{f(a)}{g(a+c)}x^c + \frac{f(a+c)\,f(a)}{g(a+2c)\,g(a+c)}x^{2c} - \text{etc.} \right\} . \quad (3.8)$$

There are as many series as there are roots of the equation

$$g(m_1) = 0$$

and in the case in which we are interested this number

s two, and the complete solution will be the sum of 3·8 and another series like it with another arbitrary constant nstead of A_1 and beginning with another power of x han a.

Certain complications may occur in the series, for we may have a zero factor in the numerator, denominator or in both simultaneously in the coefficients of the series. In some cases the two series are infinite.

We have no need to go into these cases in detail, for provided that the series are convergent we have only two points to bear in mind; the first is that if one of the series terminates and therefore contains only a finite number of terms this corresponds to the occurrence of a line spectrum and the appropriate values of E are the corresponding energy levels.

If the series is infinite this corresponds to a continuous spectrum.

The condition that the series shall terminate is that $f(a)$, $f(a + c)$ or one of the functions occurring in the numerator shall vanish.

We will first examine the condition for convergence.

A series $U_1 + U_2 + U_3 + \ldots$ is convergent provided that after some term in the series the ratio of one term to the next is greater than unity, i.e. $\left|\frac{U_n}{U_{n+1}}\right| > 1$ for values of n greater than some particular value.

If we apply this to the series (3·8) we find that the condition is:

$$\left|\frac{g(a + \overline{n+1}c)}{f(a + nc)} \cdot \frac{1}{x^c}\right| > 1.$$

Thus remembering the form of g and f we find, when n is large, that

$$\left|\frac{b_2}{a_2} \cdot \frac{1}{x^c}\right| > 1.$$

or $x^c < \left|\frac{b_2}{a_2}\right|$.

It will be noted that if a_2 is zero, the term a_2x^{k+2} not occurring in the equation, the series is convergent for all finite values of x no matter how large.

In the cases we have to consider the term a_2x^{k+2} does not occur, so that no difficulty arises with regard to convergence.

The equation (3·3) can now be written thus :

$$\frac{d^2V}{dx^2} + \left(a_1x^{p+1} + \frac{b_1}{x}\right)\frac{dV}{dx} + \left(a_0x^p + \frac{b_0}{x^2}\right)V = 0 \quad (3·9)$$

where slight changes have been made in the use of the constants, p being written for $(k - l - 2)$.

We have considered the method of solution of (3·9) which we have stated is the form on which the solution of the wave equation ultimately depends. We must consider how the wave equation can be reduced to this known form.

We begin in general with (3·2) and make the substitution $y = FV$, and since V is a finite series we require that F should approach zero as x approaches infinity in order to satisfy our boundary condition.

On substituting for y we obtain :

$$\frac{d^2V}{dx^2} + \left(\frac{2}{F}\frac{dF}{dx} + P\right)\frac{dV}{dx} + \left(\frac{1}{F}\frac{d^2F}{dx^2} + \frac{P}{F}\frac{dF}{dx} + Q\right)V = 0 \quad (3·10)$$

In order that this shall be of the form (3·9) we must choose F to satisfy the conditions :

$$\frac{2F'}{F} + P = a_1x^{p+1} + \frac{b_1}{x}$$

$$\frac{F''}{F} + \frac{PF'}{F} + Q = a_0x^p + \frac{b_0}{x^2},$$

writing F' and F'' for $\frac{dF}{dx}$ and $\frac{d^2F}{dx^2}$ respectively.

Now P and Q are polynomials, so that $\frac{F'}{F}$ and $\frac{F''}{F}$ are also polynomials, and consequently the differential

coefficients must be divisible by F. We write, therefore, $F = e^{\phi}$ and have to choose ϕ to satisfy

$$\left.\begin{aligned} 2\phi' + P &= a_1 x^{p+1} + \frac{b_1}{x} \\ \phi'' + \phi'^2 + P\phi' + Q &= a_0 x^p + \frac{b_0}{x^2} \end{aligned}\right\}. \quad (3{\cdot}11)$$

ϕ' is thus a polynomial in x, and if ϕ is a polynomial function of x with negative coefficients e^{ϕ} will approach zero as x approaches infinity, and our condition at infinity is satisfied.

We need not proceed farther to express F in terms of x in the general case, we shall have examples of this in the cases we have to consider.

The Planck Oscillator

The first example is one of historical interest because it is that to which Planck first applied his original quantum postulate.

It is an excellent example of the application of the principles of the new mechanics.

The oscillator is a particle vibrating under a restoring force proportional to the displacement and executes simple harmonic vibrations.

Suppose that the particle vibrates along the axis of x and that at any instant it is displaced to a point distant x from this mean position. If the restoring force is denoted by kx the equation of motion is :

$$m\ddot{x} = -kx \quad . \quad . \quad . \quad . \quad (3{\cdot}12)$$

The solution of this equation is :

$$x = A \sin (2\pi\nu t + \alpha),$$

where A and α are constants and $4\pi^2\nu^2 = \frac{k}{m}$, ν denoting the frequency of the vibrations.

The potential energy at x is $\frac{1}{2}kx^2 = 2m\pi^2\nu^2x^2$.

The energy equation is :

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \mathrm{E}$$

and writing $p = m\dot{x}$ we have :

$$\frac{1}{2m}p^2 + 2m\pi^2\nu^2x^2 = \mathrm{E}.$$

Thus according to the rule given at the end of the last chapter the wave equation is :

$$\frac{1}{2m}\left(\frac{h}{2\pi i}\frac{d}{dx}\right)^2\psi + 2m\pi^2\nu^2x^2\psi = \mathrm{E}\psi$$

or

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h}(\mathrm{E} - 2m\pi^2\nu^2x^2)\psi = 0.$$

If we write $a = \dfrac{8\pi^2m\mathrm{E}}{h^2}$ and $b = \dfrac{16\pi^4m^2\nu^2}{h^2}$ for convenience this equation is more simply written :

$$\frac{d^2\psi}{dx^2} + (a - bx^2)\psi = 0 \quad . \quad . \quad . \quad (3{\cdot}13)$$

This equation is of the type (3·2) and we can convert it to the type we have been considering by means of equations (3·11).

In this special case $\mathrm{P} = 0$, $\mathrm{Q} = a - bx^2$, and it will be noted that if we write $p = 0$, $b_1 = 0$, $a_1 = -2b^{\frac{1}{2}}$ we have $\phi = -\frac{1}{2}b^{\frac{1}{2}}x^2$. The negative sign is required to make ψ remain finite and approach zero when x approaches infinity.

The second equation in (3·11) may be satisfied by choosing the right values for a_0 and b_0, which are $a_0 = a - b^{\frac{1}{2}}$ and $b_0 = 0$. It is simplest to obtain the transformed equation by making the substitution directly in (3·13) rather than to use the general case ; thus on substituting $\psi = e^{-\frac{1}{2}b^{\frac{1}{2}}x^2}\mathrm{V}$ we find :

$$\frac{d^2\mathrm{V}}{dx^2} - 2b^{\frac{1}{2}}x\frac{d\mathrm{V}}{dx} + (a - b^{\frac{1}{2}})\mathrm{V} = 0 \quad . \quad (3{\cdot}14)$$

which is of the form (3·9), and of which we have studied the solution.

For the sake of convenience we will write (3·14) thus :

$$\frac{d^2V}{dx^2} - cx\frac{dV}{dx} + eV = 0$$

and make the substitution :

$$V = \Sigma A_1 x m_1$$

and it then follows that the values of the lowest index m_1 are either zero or unity, and that each series advances in powers increasing by 2.

The constant coefficients are given in terms of the lowest by

$$A_2 = \frac{(cm_1 - e)}{(m_1 + 2)(m_1 + 1)} A_1,$$

$$A_3 = \frac{(cm_1 + 2 - e)(cm_1 - e)}{(m_1 + 4)(m_1 + 3)(m_1 + 2)(m_1 + 1)} A_1, \text{ etc.}$$

There is no difficulty in either of the series which arise when the values $m_1 = 0$ and $m_1 = 1$ are introduced, for no zero factor occurs anywhere in the denominators.

We have to decide whether either or both of the series will terminate, and this, as we have mentioned, corresponds to the question in physics as to whether this oscillator will give rise to line spectra or continuous spectra or to one of each.

The series will terminate if one of the factors in the numerator vanishes, for if in advancing in the series any A vanishes then all succeeding A's vanish by the above relations.

Now the factors in the numerator are of the form :

$$c(m_1 + 2n) - e$$

where n is an integer.

This will vanish if

$$n = \frac{e - m_1 c}{2c}$$

i.e., if the quantity on the right is an integer.

If we take the case when $m_1 = 0$ we see that this requires $\frac{e}{2c}$ to be an integer.

Now if we refer to the quantities for which e and c stand we find that $\frac{e}{c} = \frac{a}{2b^2} - \frac{1}{2}$

$$= \frac{E}{h\nu} - \frac{1}{2}.$$

Thus the series terminates if $E = (2n + \frac{1}{2})h\nu$.

Again, if $m_1 = 1$ the series will terminate if

$$c(1 + 2n) - e = 0,$$

i.e. if $$2n + 1 = \frac{e}{c} = \frac{E}{h\nu} - \frac{1}{2}.$$

i.e. if $$E = (2n + \tfrac{3}{2})h\nu$$

$$= (\overline{2n + 1} + \tfrac{1}{2})h\nu.$$

From the two results the possible values of E may be written :

$$E = (n + \tfrac{1}{2})h\nu$$

where n denotes any integer, odd or even.

These values of E may be said to be those physically possible. There is nothing in our derivation to show that others may not occur, but apparently this is the case, for it is well known that the assumption that the oscillator possesses these particular energy values has cleared up a very great difficulty in the problem of radiation.

The occurrence of the $\frac{1}{2}h\nu$ is noteworthy, for the older view was that $E = nh\nu$, while half-quantum numbers had to be recognized ; the half-quantum number occurs here quite naturally as a result of the theory.

The Rotator

In the next example we take the case of a body rotating about a fixed axis. Rotators of this type are considered in the theory of band spectra, although in this case the problem is not so simple as that we shall

consider, for the axis in our case is supposed to be fixed in space, while in the band-spectrum problem it is free.

Our example has no physical application, but it is the simplest of all wave mechanics problems and it illustrates a case in which it is difficult to form any picture of actual waves. It is, in fact, a generalized dynamical wave problem with one degree of freedom

There is no potential energy, and if I denote the moment of inertia and θ the angular co-ordinate, we have for the energy equation simply :

$$\tfrac{1}{2}\mathrm{I}\dot{\theta}^2 = \mathrm{E}.$$

Writing $p = \dfrac{\partial \mathrm{T}}{\partial \dot{\theta}} = \mathrm{I}\dot{\theta}$ we deduce :

$$\frac{1}{2\mathrm{I}}p^2 = \mathrm{E}.$$

Thus the wave equation is :

$$\frac{1}{2\mathrm{I}}\left(\frac{h}{2\pi i}\frac{d}{d\theta}\right)^2\psi = \mathrm{E}\psi$$

or

$$\frac{d^2\psi}{d\theta^2} + \frac{8\pi^2\mathrm{I}\mathrm{E}\psi}{h^2} = 0.$$

The solution is :

$$\psi = \mathrm{A}\sin(k\theta + \alpha)$$

where $k^2 = \dfrac{8\pi^2\mathrm{I}\mathrm{E}}{h^2}$, and A and α are constants.

If ψ is to be single valued it must have the same value when θ changes by 2π, and consequently it is periodic in θ.

Thus k is an integer, n, and we find :

$$\mathrm{E} = \frac{h^2}{8\pi^2\mathrm{I}}n^2.$$

The form of this simple result is very suggestive, for it is a reminder of the band-spectra energy formulæ, but on considering the case for the moving axis the energy values are :

$$E = \frac{n(n+1)}{8\pi^2 I} h^2$$

which is a surprising result, for not only does it give the actual energy values, but it gives immediately the half-quantum numbers which have been most difficult to explain in this connection.

The formula for E can be written :

$$E = \frac{h^2}{8\pi^2 I}\{(n+\tfrac{1}{2})^2 - \tfrac{1}{4}\}$$

and the differences corresponding to two energy levels are thus of the form :

$$E' - E'' = \frac{h^2}{8\pi^2 I}\{(n'+\tfrac{1}{2})^2 - (n''+\tfrac{1}{2})^2\}$$

The Line Spectrum of Hydrogen

Another problem of importance is given by the hydrogen atom, and we now undertake the determination of the energy levels in this case. This is treated in the same way as the others, the only difference being that the details of the calculations are a little longer.

The mechanical system in this case consists of a central positive charge e, and an electronic charge of equal magnitude at a distance r from it.

We may derive the wave equation by the symbolical method we have just employed, but we will do so directly in this case in order to illustrate the other way of determining the equation in a special problem.

In the equation (2·25) we have to substitute $V = -\frac{e^2}{r}$ in order to deal with our problem, and the wave equation is consequently :

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r}\right)\psi = 0 \ . \quad (3{\cdot}15)$$

On account of the occurrence of r we will make use of polar co-ordinates. The first three terms then assume a well-known form which we will write down without

proof since the transformation is readily made and is given in mathematical text-books.

The transformed equation is :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$
$$+\frac{8\pi^2m}{h^2}\left(\mathrm{E}+\frac{e^2}{r}\right)\psi=0 \quad . \quad . \quad . \quad (3{\cdot}16)$$

The method adopted for solving this equation is to try the substitution :

$$\psi=\mathrm{R}\Theta\Phi$$

where R is a function of r only, Θ of θ only and Φ of ϕ only. (3·16) then becomes :

$$\frac{1}{\mathrm{R}r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\mathrm{R}}{\partial r}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$
$$+\frac{1}{\Phi r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{8\pi^2m}{h^2}\left(\mathrm{E}+\frac{e^2}{r}\right)=0.$$

From the fact that R and Θ do not contain ϕ we know that the only term that could contain this co-ordinate is $\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$, but the equation shows that this term does not contain it.

Thus $\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$ is independent of r, θ or ϕ, and is thus a constant. Let this constant be written $-p^2$.

The equation now becomes :

$$\frac{1}{\mathrm{R}r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\mathrm{R}}{\partial r}\right)+\frac{1}{r^2}\left\{\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{p^2}{\sin^2\theta}\right\}$$
$$+\frac{8\pi^2m}{h^2}\left(\mathrm{E}+\frac{e^2}{r}\right)=0.$$

By the same argument as the above it follows that the quantity in brackets { } is a constant. It is usual to write this constant in the form $-n(n+1)$ where n may be any integer or may have the value zero. This limitation upon the form of the constant is necessary

in order that Θ may be a single-valued function of θ. The equation now takes the form :

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left\{\frac{8\pi^2 mE}{h^2}+\frac{8\pi^2 me^2}{r}-\frac{n(n+1)}{r^2}\right\}R=0 \quad (3\cdot16)^1$$

These considerations are an outline of a general treatment which precedes one method of beginning the study of spherical and tesseral harmonics. There has so far been nothing which is especially characteristic of our problem. The equation (3·16) is, however, peculiar to our special case in that it contains the term in $\frac{1}{r}$. It is thus from the form of R that we shall derive information of interest in this example.

The actual value of ψ is the product of R, Θ and Φ. The last of these is equal to $(A_p \sin p\phi + B_p \cos p\phi)$ where A_p and B_p are constants and p may have any integral value.

Θ is the solution of the equation obtained by equating the expression in the brackets $\{\ \}$ to $-n(n+1)$. It is a rather complicated function, but very well known to the mathematician and widely applied in physics. We shall not require these functions, and so pass on to find what limitations (3·16) will place upon E. This equation is of the form we have discussed, and it will be found by the above methods that the appropriate substitution is :

$$R = e^{\frac{1}{2}a_1 r}V \quad . \quad . \quad . \quad . \quad (3\cdot17)$$

and that 3·16 then gives :

$$\frac{d^2V}{dr^2}+\left(a_1+\frac{b_1}{r}\right)\frac{dV}{dr}+\left(\frac{a_0}{r}+\frac{b_0}{r^2}\right)V=0 \quad (3\cdot18)$$

where

$$\left.\begin{aligned} \tfrac{1}{4}a_1^2 &= -\frac{8\pi^2 mE}{h^2} \\ a_0 &= a_1+\frac{8\pi^2 me^2}{h^2} \\ b_1 &= 2 \\ b_0 &= -n(n+1) \end{aligned}\right\} \quad . \quad . \quad (3\cdot19)$$

The equation (3·18) may be written :

$$\left(a_1 r\frac{d}{dr} + a_0\right)\mathrm{V} + \frac{1}{r}\left\{\left(r\frac{d}{dr}\right)^2 + r\frac{d}{dr} + b_0\right\}\mathrm{V} = 0$$

or writing $f(z) = a_1 z + a_0$

$$g(z) = z^2 + z + b_0$$

we have

$$f\left(r\frac{d}{dr}\right)\mathrm{V} + \frac{1}{r}g\left(r\frac{d}{dr}\right)\mathrm{V} = 0$$

which is of the form (3·6).

The solution is thus given by the sum of two series of the form (3·8) where the appropriate values of a are the roots of the equation :

$$g(m_1) = 0$$

i.e. $$m_1^2 + m_1 + b_0 = 0.$$

On making the substitution for b_0 given by (3·19) we find that the two values are n and $-(n+1)$, so that the series begin with r^n and $r^{-(n+1)}$.

If we consider the series beginning with $r^{-(n+1)}$ we see that it will terminate if $f(a)$, $f(a+c)$, or any such term as $f(a+sc)$, vanish, where s is an integer and c is unity in our problem. Thus, since $f(z) = a_1 z + a_0$, we have to examine under what conditions :

$$a_1(a + sc) + a_0$$

will become zero.

Then writing $a = -(n+1)$, $c = 1$, we find that this requires that

$$-n - 1 + s = -\frac{a_0}{a_1}$$

or $$s - n = \frac{a_1 - a_0}{a_1}.$$

By making use of the values of a_1 and a_0 given above, we find that the condition is :

$$(s-n)^2 = -\frac{2\pi^2 me^4}{\mathrm{E}h^2}.$$

Thus since s and n are integers, we may denote $(s - n)$ by an integer τ and write :

$$E = -\frac{2\pi^2 me^4}{h^2\tau^2}.$$

It follows that the energy levels in the line series of the hydrogen spectrum are given by this series of values of E. These are, of course, the Bohr energy values.

In order that R, and consequently ψ, may become zero at infinity, the value of a_1 in the exponential factor $e^{ia_1 r}$ must be negative and real. Thus the value of E must be negative (3·19), and the reader should compare this with Shroedinger's discussion in his first paper (4) *Ann. d. Physik.*, Vol. 79, 1926.

These values of E account for the Balmer series in a manner familiar in the old quantum theory, but no explanation is given for the fine structure of the lines in this series in the account just given.

The Fine Structure of the Lines in the Hydrogen Spectrum

In order to take into account the variation of mass with speed which becomes important at the speeds occurring in atomic problems, we must apply (2·22) directly without the approximation which leads to (2·25). From (2·24)

$$G^2 = \frac{1}{c^2}\left(E + \frac{e^2}{r}\right)^2 + 2m_0\left(E + \frac{e^2}{r}\right)$$

and the wave equation is thus :

$$\nabla^2\psi + \frac{4\pi^2}{h^2}\left\{\left(\frac{E^2}{c^2} + 2m_0 E\right) + \frac{2}{r}\left(m_0 e^2 + \frac{Ee^2}{c^2}\right) + \frac{e^4}{c^2 r^2}\right\}\psi = 0.$$

If this equation is treated in the same way as (3·15) we obtain the following equation in R instead of (3·16).

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left\{a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} - \frac{n(n+1)}{r^2}\right\}R = 0 \quad (3{\cdot}20)$$

where $a_0 = \frac{4\pi^2}{h^2}\left(\frac{E^2}{c^2} + 2m_0E\right)$, $a_1 = \frac{8\pi^2}{h^2}\left(m_0e^2 + \frac{Ee^2}{c^2}\right)$,

$$a_2 = \frac{4\pi^2e^4}{h^2c^2} \quad (3{\cdot}21)$$

Make the substitution $R = e^{\frac{1}{2}kr}V$ and we obtain the equation :

$$\frac{d^2V}{dr^2} + \left(k + \frac{2}{r}\right)\frac{dV}{dr} + \left(\frac{k^2}{4} + a_0 + \frac{a_1 + k}{r} + \frac{b^2}{r^2}\right)V = 0,$$

where $b^2 = a_2 - n(n+1)$, and k is real and negative.

Let k be chosen so that $\left(\frac{k^2}{4} + a_0\right) = 0$, and we have an equation of the type considered which can be written

$$f\left(r\frac{d}{dr}\right)V + \frac{1}{r}g\left(r\frac{d}{dr}\right)V = 0,$$

where $f(z) = kz + a_1 + k$

and $g(z) = z^2 + z + b^2$

with $k^2 = -4a_0 = -\frac{16\pi^2}{h^2}\left(\frac{E^2}{c^2} + 2m_0E\right)$, so that $\left(\frac{E^2}{c^2} + 2m_0E\right)$ is a negative quantity and k must be chosen with the negative sign before the square root.

The indices of the first terms in the series are the roots of the equation :

$$g(m_1) = 0$$

i.e.

$$m_1^2 + m_1 + b^2 = 0$$

and the series corresponding to a root a of this equation terminates when $f(a + sc) = 0$, where again $c = 1$ and s is an integer. Thus termination of the series requires :

$$k(a + s) + a_1 + k = 0,$$

where

$$a^2 + a + b^2 = 0.$$

In order to find the appropriate values of E we have to refer to the values of the quantities in these two equations.

From the second it is easy to obtain :

$$(a + \tfrac{1}{2})^2 = (n + \tfrac{1}{2})^2 - l^2$$

where for convenience we write : $l = \sqrt{a_2} = \dfrac{2\pi e^2}{hc}$.

From the first :

$$s + a = -1 - \frac{a_1}{k}.$$

Hence $$a + \tfrac{1}{2} = -s - \tfrac{1}{2} - \frac{l(\mathrm{E} + m_0c^2)}{\sqrt{-(\mathrm{E}^2 + 2m_0c^2\mathrm{E})}}$$

from which we may deduce :

$$\mathrm{E} + m_0c^2 = m_0c^2\left[1 - \frac{l^2}{\{s + \tfrac{1}{2} + \sqrt{(n+\tfrac{1}{2})^2 - l^2}\}^2 + l^2}\right]^{-\frac{1}{2}} \quad (3{\cdot}22)$$

The formula for E is almost identical with that given by Sommerfeld for the relativity Kepler motion in the hydrogen atom (cf. *Atombau u. Spektrallinien*, 3rd Edn., p. 577 (24)). The chief difference is that his integers n' and n are replaced by $(s + \frac{1}{2})$ and $(n + \frac{1}{2})$.

Thus our integer s corresponds to the radial quantum number n' and n, which is introduced in connection with the function Θ, is the azimuthal quantum number.

In view of the discussion on the origin of the fine structure of the lines in the hydrogen spectrum, this result is interesting. It shows that a fine structure is anticipated in the new mechanics on account of the relativity correction. It is apparently of no purpose to discuss the accuracy of the formula just obtained in order to compare it with that of the older quantum theory. It appears that the multiplicity of spectral lines, both in the optical and X-ray regions, is to be closely associated with the magnetic state of the atom and with the abnormal Zeeman effect and the sorting out of what is purely a relativity effect is a very complex problem.

The Rotator with a Free Axis

The problem now to be considered is that which has been mentioned in the second example, it is that of a rotator about an axis which is free. As an illustration of the system we may imagine two spheres at a fixed distance apart rotating about an axis bisecting perpendicularly the line joining their centres. We have a dumbell with the two spheres lying on a plane and rotating about a normal to the plane. In the former case which we considered this plane was fixed in space, i.e. the axis of rotation was fixed, we now consider the case in which the plane can move, the axis being free to take up any position. We do not contemplate any motion of the dumbell about the line of centres.

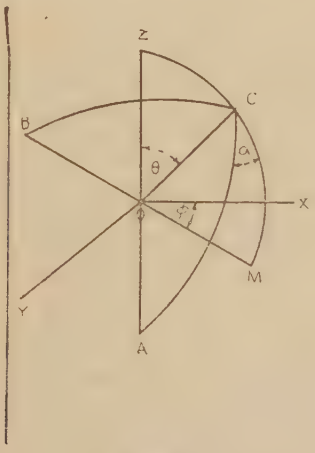

Fig. 8.

Let OC denote the line of centres of the system, so that it is the axis about which there is no rotation. Let the position of OC in space be described in the usual way by θ and ϕ and let ZOCM be the plane on which the system rests.

Let OA and OB be any two lines perpendicular to each other and to OC.

For convenience we will suppose that OA, OB and OC are of unit length and let AC and BC be joined by circular arcs. Since OC is moving freely the system will have a rotation which may be considered as having components about OA and OB, these components being denoted by ω_1 and ω_2. It is, of course, well known that a rotation about an axis may be treated as a vector, and since there is no rotation about OC the axis of rotation must lie in the plane OAB.

Let us suppose that the system is such that the moment of inertia about any line in the plane OAB is I. Thus since the angular velocities about OA and OB are ω_1 and ω_2, the kinetic energy of the system is :

$$\tfrac{1}{2}\mathrm{I}(\omega_1^2 + \omega_2^2).$$

We must express ω_1 and ω_2 in terms of θ and ϕ.

Let α denote the angle between the plane OAC and OCM, then the velocity of C along ZCM is $\dot{\theta}$ and perpendicular to the plane ZOM, $\sin\theta.\dot{\phi}$. Expressed in terms of ω_1 and ω_2 the velocity of C is ω_1 along the arc CB and ω_2 along AC, where ω_1 and ω_2 are regarded as right-handed twists about OA and OB.

Thus the velocity of C along ZCM is

$$-(\omega_1 \sin\alpha + \omega_2 \cos\alpha)$$

and perpendicular to ZOM $(\omega_1 \cos\alpha - \omega_2 \sin\alpha)$.

Thus $$\dot{\theta} = -(\omega_1 \sin\alpha + \omega_2 \cos\alpha)$$

and $$\sin\theta.\dot{\phi} = \omega_1 \cos\alpha - \omega_2 \sin\alpha.$$

From these equations we find that the above expression for the kinetic energy is :

$$\mathrm{T} = \tfrac{1}{2}\mathrm{I}(\dot{\theta}^2 + \sin^2\theta.\dot{\phi}^2)$$

or writing $$p_1 = \frac{\partial \mathrm{T}}{\partial \dot{\theta}}, \quad p_2 = \frac{\partial \mathrm{T}}{\partial \dot{\phi}}$$

$$\mathrm{T} = \frac{1}{2\mathrm{I}}\left(p_1^2 + \frac{p_2^2}{\sin^2\theta}\right)$$

In this case there is no potential energy, so that :

$$\frac{1}{2I}\left(p_1^2 + \frac{p_2^2}{\sin^2\theta}\right) = E.$$

This is a case in which we have two generalized co-ordinates θ and ϕ, so that our space is of two dimensions. In order to find the wave equation we must find the meaning of $\nabla^2\psi$ in this space and then express the equation :

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}E\psi = 0$$

in θ and ϕ.

It is exactly of the form which occurs for a particle moving on the surface of unit sphere, the only difference being that I replaces the mass of the particle.

Thus the method of transforming $\nabla^2\psi$ in this case is simply to place $r = 1$ in (3·16) in the first three terms, leaving for the wave equation :

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\theta^2} + \frac{8\pi^2 IE}{h^2}\psi = 0 \quad (3{\cdot}23)$$

The method of solution is similar to that employed in the problem of the hydrogen atom, and from the discussion in connection with it we see that the condition that ψ shall be single-valued in θ requires that :

$$\frac{8\pi^2 IE}{h^2} = n(n+1)$$

where n is an integer.

Thus

$$E = \frac{h^2}{8\pi^2 I}n(n+1) = \frac{h^2}{8\pi^2 I}\{(n+\tfrac{1}{2})^2 - \tfrac{1}{4}\} \quad . \quad (3{\cdot}24)$$

which gives the result quoted in connection with the problem of the rigid rotator about the free axis. It is interesting to see how the integer values and the half-quantum numbers come in by the application of a very well-known result in mathematics which we regard as part of the properties of the legitimate stage of classical physics.

The Occurrence of Half-Quantum Numbers

Half-quantum numbers occur naturally in the energy values according to our theory and merit no special consideration from this point of view. They have been a source of difficulty and the subject of much discussion in the old quantum theory, and we will end this chapter with an examination of the conditions under which they occur or are absent.

We have seen that the solution of the problem depends finally on the equation:

$$\frac{d^2V}{dx^2} + \left(a_1x^{p+1} + \frac{b_1}{x}\right)\frac{dV}{dx} + \left(a_0x^p + \frac{b_0}{x^2}\right)V = 0 \quad (3\cdot9)$$

This may be written in the form:

$$f\left(x\frac{d}{dx}\right)V + \frac{1}{x^{p+2}}g\left(x\frac{d}{dx}\right)V = 0$$

where
$$f(z) = a_1z + a_0$$
$$g(z) = z^2 + (b_1 - 1)z + b_0$$

Let a denote a root of $g(z) = 0$, and suppose we take

$$a = \frac{1 - b_1}{2} - \tfrac{1}{2}\sqrt{(b_1 - 1)^2 - 4b_0}$$

The series corresponding to this root begins with x^a and advances with the indices increasing by $(p + 2)$ from term to term.

The series terminates if:

$$f(a + sc) = 0,$$

where c is written for $(p + 2)$ and s is an integer.

This condition is:

$$a_1(a + sc) + a_0 = 0$$

or
$$sc = -a - \frac{a_0}{a_1}$$
$$= -\tfrac{1}{2} + \frac{b_1}{2} - \frac{a_0}{a_1} + \tfrac{1}{2}\sqrt{(b_1 - 1)^2 - 4b_0},$$

and there is a similar condition for the other root con

aining a negative sign before the square root in the ast equation.

The occurrence of the first term on the right is note-vorthy, for it is responsible for the half-quantum number.

Whole numbers occur in the case where $b_0 = 0$, since he last term contains $\frac{1}{2}$.

This has been illustrated in the problem of the ydrogen spectrum where $c = 1$ and $b_0 = 0$ or $\neq 0$ ccording as we take the mass of the electron as con-tant or as a function of the speed.

Thus we must regard the occurrence of half-quantum umbers as merely dependent on the values of the con-tants and not as indicating the existence of any principle vhich has been overlooked in founding the quantum heory.

The fraction $\frac{1}{2}$ will be reduced to $\frac{1}{4}$ in a problem in vhich $c = 2$, and for different values of c it will be eplaced by other values.

CHAPTER IV

EVIDENCE FOR THE EXISTENCE OF MECHANICAL WAVES

THE foregoing chapters have shown us that if we adopt a certain equation and apply it to problems in mechanics we are able to calculate certain values for the energy which agree with observation in atomic physics. In this respect the equation plays the part of a correlator of known results, we turn to it also in the search for new results to seek guidance for further experiments.

Mathematics has been compared to a librarian whose duty it is to correlate his works and to point to empty places on his shelves where new volumes are required.

The new equation appears in the form of a wave equation, and it may be that this form is merely accidental, there may actually be no waves. The function ψ may have no physical meaning, but in the light of the history of the growth of mathematics and physics it would be a neglect of a very clear indication that information is required on this point.

We must then try to find the physical meaning of ψ and search in old and new experiments for evidence on the possible existence of waves associated with matter which is what our theory suggests. We must also carry out work of a less original character to determine whether formulæ for the energy values agree with experiment as well as, or better than, those obtained by other methods.

In this chapter we propose to consider evidence at our disposal and experiments that have been performed which suggest that such waves actually exist.

These experiments concern the scattering of electrons and the results obtained are, from our point of view, both of a qualitative and quantitative character. Some of the results of electron scattering suggest, at any rate qualitatively, a resemblance to diffraction. In order to test the process quantitatively we have to look for waves of a wave-length equal to λ where :

$$\lambda = \frac{U}{\nu} = \frac{W}{\nu G} = \frac{h}{G} \quad . \quad . \quad . \quad (4{\cdot}1)$$

by the equation (2·3).

From our point of view the discovery of diffraction-like phenomena in connection with electrons is very apt, for we have seen that we might anticipate the same change in micro-mechanics that we have in passing from geometrical to physical optics.

We begin with an experiment by C. Davisson and C. H. Kunsman described in the *Physical Review* (Vol. 22, p. 242, 1923).

These experimenters studied the scattering of a beam of electrons by platinum and magnesium. At the time when the results were obtained a theoretical interpretation was given with which we are not concerned. The explanation was offered before the new mechanics had been considered and it is not based on the theory we are discussing.

The experiment consisted in causing a directed beam of electrons to fall on a platinum or magnesium surface and in examining the electrons scattered at various angles.

The electrons originated at a tungsten filament and were accelerated by the application of differences of potential up to 1,000 volts. They were restricted to a narrow beam by means of a 'chute' and struck the metal surface at an angle of incidence of 45°. A collecting box was placed to receive the scattered electrons and could be rotated to receive them in directions making from 15° to 135° with the primary beam.

The current in the primary beam was measured, as

also that in the scattered beam, the currents being represented by i_p and i_s. The value of the ratio $\frac{i_s}{i_p}$ was determined in the different directions.

The order of the currents may be represented thus:

$$3 \times 10^{-6} \leqslant i_p \leqslant 9 \times 10^{-5} \text{ amp.}$$

and the order of the ratio thus: $10^{-5} \leqslant \frac{i_s}{i_p} \leqslant 10^{-3}$.

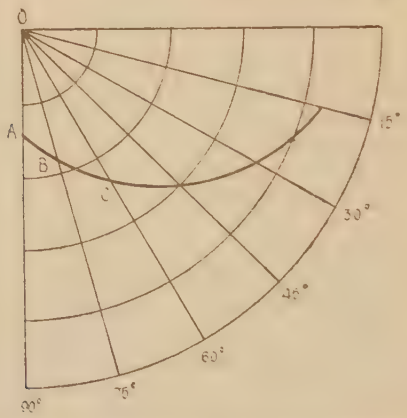

FIG. 9.—Scattering of 150-volt electrons.

The curve in Fig. 9 is drawn so that the radius vector in any direction measures the ratio corresponding to that direction; thus OA measures the ratio in the direction at 90° to the primary beam.

In this case the primary beam consisted of electrons accelerated by a potential drop of 150 volts, the scattered electrons being collected against a retarding potential of 135 volts so that all those reaching the collector had not lost more than 10 per cent. of their initial energy. In this way it was made certain that the electrons from the metal surface had not arisen as a result of any secondary effects of the bombardment.

The curve indicates the approach to a maximum scattering in the neighbourhood of the direction of the primary beam.

In Fig. 10 the results are shown for the case when the primary beam was accelerated by a potential difference of 500 volts.

The interest is in the occurrence of maxima at the points A and B showing a more intense scattering in the directions OA and OB.

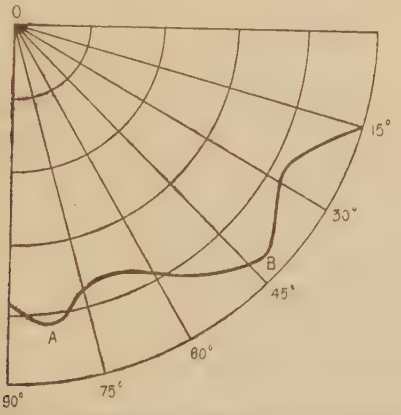

FIG. 10.—Scattering of 500-volt electrons.

This bears at least a superficial resemblance to the phenomenon of diffraction of light rays where maximum of amounts of energy are diffracted along certain directions.

The third of this series of diagrams (Fig. 11) illustrates the case when the acceleration was 1,000 volts. In this case maxima occur again but they are displaced from the directions of the second experiment suggesting a change similar to that occurring in diffraction when there is a change in wave-length.

In these experiments there is a change in voltage, and consequently in the momentum of the electrons as

they reach the deflecting surface; this, from (4·1), means a change in wave-length of the mechanical waves associated with the electrons.

Results of a similar character have been obtained by Davisson and Germer more recently (*Nature*, 119, p. 558) by reflection from nickel crystals showing the same qualitative agreement with the wave theory.

The scattering of electrons by helium has been examined by Dymond (*Physical Review*, Vol. 29, p. 433,

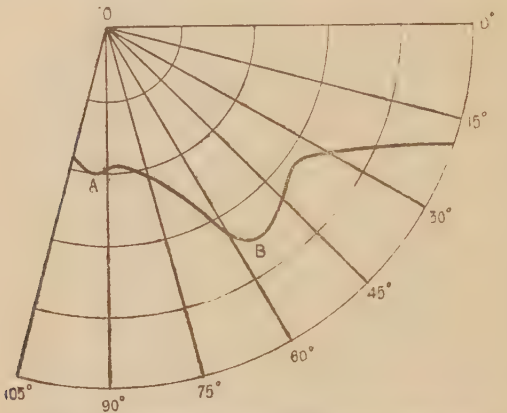

FIG. 11.—Scattering of 1,000-volt electrons.

1927). In these experiments accelerated electrons were allowed to fall on helium and the scattering observed at various inclinations to the primary beam. The electrons passed through two slits after scattering, and were then deflected by a magnetic field into a collector. It was thus possible to study the velocities after scattering, but we are not concerned with this part of the experiment. The magnetic field was adjusted so that electrons of a particular velocity were collected, the amount of energy lost in the process of scattering being thus known.

The curves shown in Figs. 12, 13 and 14 show the

ame displacement of the maxima with the different voltages as in the previous cases.

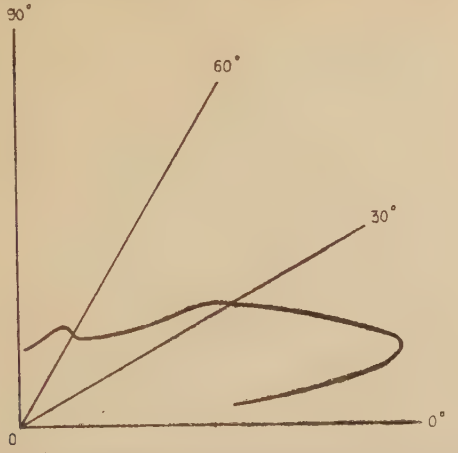

FIG. 12.—Scattering of 50-volt electrons by helium.

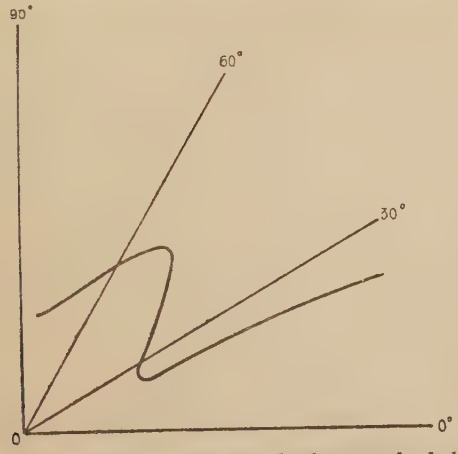

FIG. 13.—Scattering of 100-volt electrons by helium.

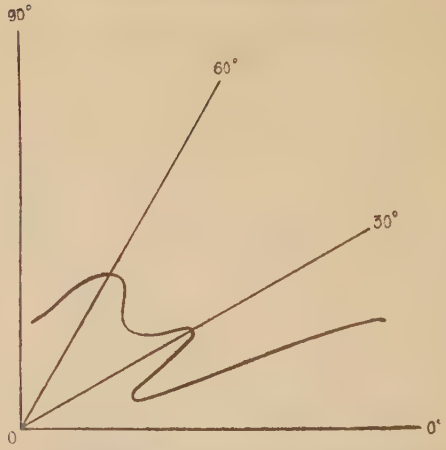

FIG. 14.—Scattering of 200-volt electrons by helium.

We now pass on to consider some experiments which permit of a quantitative examination.

For the purpose of making a calculation we have to form some theory underlying the phenomenon we are investigating and it is, of course, very much to the point to determine the wave-length of the mechanical waves concerned. In the experiments electrons were examined which were subjected to a potential drop of the order of 25,000 volts. We will first determine the wave-length in this case, making the calculation in the simple way without any correction for the variation of mass with speed.

If m denote the mass of the electron and v the velocity acquired in a field corresponding to a potential difference P absolute units,

$$Pe = \tfrac{1}{2}mv^2.$$

or

$$v = \sqrt{\frac{2Pe}{m}}.$$

The wave-length is thus, $\lambda = \dfrac{h}{\sqrt{2mPe}}$ and on substi-

uting the appropriate values we obtain $\cdot 78 \times 10^{-9}$ cm. r ·078 Ångströms. This is a wave-length equal to hat of very hard X-rays.

For voltages of the order 100 such as those considered bove the wave-length is 1·22 Ångströms. We shall hus expect the mechanical waves to behave in a similar ray to X-rays.

This comparison forms the basis of the experiments e now describe and of the calculations made from the esults obtained.

The wave character of X-rays was demonstrated in most striking and decisive manner by Laue, and ir William and W. L. Bragg by their study of the eflection of X-rays by crystals, and recently experiments f a similar nature have been carried out by Professor . P. Thomson and others working with him on the ehaviour of cathode rays and thin films of various naterials. The experiments resemble that associated ith the names of Debye, Hull and Sherrer in which -rays were reflected at the surfaces of a number of mall crystals arranged at random. The resulting attern on a photographic plate shows a definite sym- netry if certain crystal directions predominate.

Thomson caused a beam of cathode rays, made pproximately homogeneous, to pass normally through very thin film. The rays were then received on a hotographic plate and the pattern studied.

The apparatus is of a simple character, the rays being enerated by an induction coil and caused to pass through fine tube, then through the film to be intercepted by plate after travelling a distance of about 32·5 cm. On developing the plate a symmetrical pattern about central spot was observed similar to that observed ith X-rays.

In regarding this as evidence in favour of the presence f mechanical waves we must be convinced that particles trike the plate and produce the pattern and that the lm is essential.

It is conceivable that the cathode particles produce

rays and that these give the wave pattern, but this can be decided by generating a magnetic field between the film and plate. It was found that the pattern moved as a whole under the influence of the field as if the beam arriving at the plate consisted of charged particles all with the same velocity. This differentiated the beam from one consisting of rays such as X-rays and it was also a test for the homogeneity of the beam.

With regard to the other point it need only be stated that on removing the film only the central spot appeared on the plate and no pattern.

The first striking thing about this experiment is the qualitative agreement with our theory.

To appreciate the quantitative agreement we must turn to Bragg's theory of crystal reflection. According to this theory a crystal consists of atoms arranged in a definite way, each atom forming a centre at which the rays are reflected; from this it follows that we may regard a crystal as made up of reflecting planes oriented in certain directions. These planes pass

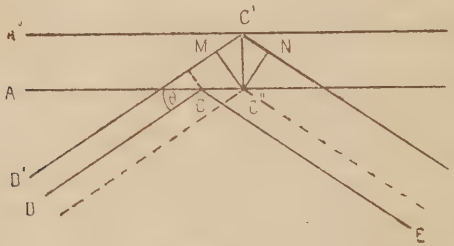

Fig. 15.

through the atoms and on account of the symmetry in the crystal structure sets of parallel planes corresponding to the different layers of atoms occur; for example, if we think of the atoms arranged at the corners of a cube reflecting planes will correspond to the sides and diagonal planes.

Let AB and A′B′ denote two consecutive parallel

planes and let two rays be incident at C and C′ and be reflected along CE and C′E′. If these rays are in the same phase they will reinforce one another and produce an intense line on a photographic plate placed to receive them.

The difference in path between them is the same as the difference between the dotted path and the second ray and this is found by drawing perpendiculars C″M and C″N to D′C′ and C′E′ respectively, to be $2d\sin\theta$ where C′C″ is normal to the planes and equal to d, while θ is the angle between the rays and the planes.

Thus the condition for reinforcement is :

$$2d\sin\theta = n\lambda, \ (n = 1, 2, 3, \text{etc.}). \quad . \quad (4{\cdot}2)$$

where d will differ according to the reflecting planes considered.

In the case of a cubic arrangement it is easy to express d in terms of the side of the cube, say a. Suppose that three directions parallel to the sides of the cubes are taken as axes, then the equation of a plane referred to these may be written : $lx + my + nz = p$, where l, m, n) denote the direction cosines of the normal to the plane and p is the perpendicular from the origin on to the plane.

Suppose that we think of a plane through the origin as one of our reflecting planes with an adjacent parallel plane at distance d.

The equation for the adjacent plane is $lx + my + nz = d$, and we could distinguish our plane by the set of direction cosines (l m n). Instead of this we take three integers (h_1 h_2 h_3) and distinguish the plane by speaking of it as the ' $h_1h_2h_3$ plane ' where the equation of the plane is thrown into the form :

$$h_1x + h_2y + h_3z = a.$$

We have, of course :

$$\frac{h_1}{l} = \frac{h_2}{m} = \frac{h_3}{n} = \frac{a}{d}$$

and since $l^2 + m^2 + n^2 = 1$ we have :

$$d = \frac{a}{\sqrt{h_1^2 + h_2^2 + h_3^2}} \quad . \quad . \quad . \quad (4{\cdot}3)$$

which gives d in terms of a from the plane indices $(h_1 h_2 h_3)$.

Suppose that a thin film containing small crystals in all possible orientations at random receives a beam of X-rays, then by (4·2) we have reinforcement if :

$$2d \sin \theta = n\lambda,$$

where d is the separation of the reflecting planes concerned and θ is the glancing angle on a small crystal.

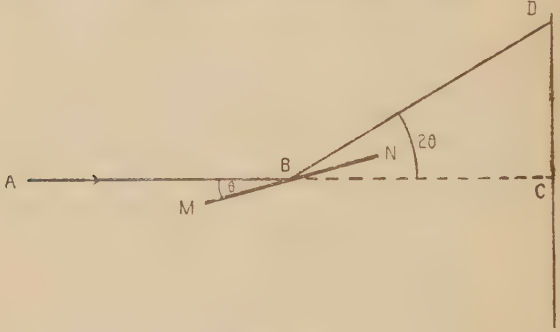

Fig. 16.

If the reflected beam is received at D on a photographic plate and the angle is small we have

$$\mathrm{CD} = 2\mathrm{L}\theta$$

where C is the point of incidence of the primary on the plate in the absence of the crystal MN and L is the distance BC, i.e. the distance from film to plate.

Thus from (4·2) we have an intense spot at D if

$$\mathrm{CD} = \frac{n\lambda \mathrm{L}}{d} \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}4)$$

If we think of B as the apex of a cone with its base

) the right of the diagram and if crystals lie so that ne planes MN are tangent to the cone at the apex, small pencil of rays along AB will fall at the same ngle on all the crystal planes giving rise to a circle n the plate of radius CD. Since we assume that the rystals are arranged at random this is what we shall get. For the same value of D we shall get a series of circles orresponding to the values of n. Other circles will be btained for different planes MN, i.e. with different alues of d, and it is possible that reflecting planes will ccur in the crystal not much inclined to MN at which eflection of this nature can occur, the different values f θ required occurring in the beam AB, which cannot e quite parallel in practice.

If the arrangement is not at random the circles may e incomplete and at any rate will show different ntensities at different parts of the ring.

In examining the experiment with cathode rays nstead of X-rays from the quantitative point of view ve have to measure the diameters of the rings formed, o find the planes which are reflecting them, calculate he wave-length from the observations and compare vith the wave-length calculated by the theory.

The method used by Thomson was essentially this, ut he made the test by calculating the constant a rom the mechanical wave-length of the cathode rays nd comparing this with the value determined in experiments on crystal structure.

It is very interesting and instructive to refer to the riginal paper on this work for the photographs of the atterns obtained. One of these is for a gold film and he rings are remarkably clear and uniform in intenity. The diameters of these rings are approximately in he ratio $1 : 1{\cdot}18 : 1{\cdot}62 : 1{\cdot}96$. We have to determine f the structure of gold crystals is such that reflectng planes occur which would produce rings with iameters in these ratios. Gold crystals are built up s a face-centred cubic lattice and for the sake of simplicity we will take three planes with indices (2, 0, 0),

(2, 2, 0) and (1, 1, 3) and find the ratios of the diameters to be expected. If we refer to (4·4) and consider the first order ring ($n = 1$) the ratios are inversely as the corresponding values of d and that according to (4·3) is in the ratio of the corresponding values of $\sqrt{h_1^2 + h_2^2 + h_3^2}$ so that in this case the ratios are

$$\begin{aligned} &\sqrt{4} : \sqrt{8} : \sqrt{11} \\ = \; &2 : 2{\cdot}83 : 3{\cdot}32 \\ = \; &1 : 1{\cdot}41 : 1{\cdot}66 \end{aligned}$$

We may write the ratios of the last three observed rings, given above, as

$$1 : 1{\cdot}39 : 1{\cdot}66$$

so that we may claim that these observed rings are accounted for. By examination of other planes it was found possible to account for the first ring.

A film of aluminium gave equally convincing evidence.

Another striking quantitative result can be obtained by observing that if D denote the diameter of a ring on the pattern $\frac{D}{\lambda}$ should be constant for the same order n. (4·4.)

Thus by varying λ and observing the value of D we can test this result.

There is no point in making a relativity correction here so that we may use the value given above, viz.:

$$\lambda = \frac{h}{\sqrt{2mPe}},$$

which shows that the theory anticipates that $D\sqrt{P}$ is a constant. Thus by measuring P, the potential applied to the cathode rays, by varying it and measuring the diameters it is easy to test the result.

A very good agreement was found, as will be seen by reference to the original paper where the results are given for aluminium and gold. The agreement here is strong evidence in favour of the wave-like behaviour of the electrons.

With regard to the determination of the value of a by this experiment, we have to state that the X-ray value for gold is $4{\cdot}065 \times 10^{-8}$ cm. and the value obtained was within about 1 per cent. of this.

Diffraction phenomena are studied in optics by passing light across straight edges, through slits and past small particles such as lycopodium powder. For waves of the length which corresponds even to slow electrons it does not seem possible to obtain diffraction fringes sufficiently separated for observation by allowing electrons to stream past edges or through slits, but it is worth while to examine the case where electrons pass through a thin film of a material not possessing crystal structure. In considering this case we shall take it to be analogous to the case of light passing through a fine powder film. This case is well known and the theory shows that for light of wave length, λ, and a particle of circular section with radius R, the first dark ring subtends at the particle an angle $2\theta_1$, where :

$$\theta_1 = 0{\cdot}61\frac{\lambda}{R},$$

or the second ring the corresponding angle is given by :

$$\theta_2 = 0{\cdot}116\frac{\lambda}{R}$$

and θ_3, θ_4, etc., can all be calculated in a similar way.

Thus if the rings are received on a photographic plate at a fixed distance from the diffracting particle and the diameters D_1, D_2, etc., are measured it follows that $\frac{D_1}{\lambda}$ is constant.

This we have seen is characteristic of the Thomson experiment, and if this halo phenomenon occurs with mechanical waves we may expect it with amorphous substances, forming films such as that described and through which a stream of cathode rays is passed.

The comparison with X-rays suggests still another case in which the ratio $\frac{D}{\lambda}$ is a constant. This has been

recently investigated numerically [1] and may form the basis of new experiments.

The appearance of the rings in Thomson's experiments calls to mind similar effects observed by Keesom and Smedt [2] in investigations on the scattering of X-rays by liquid oxygen and nitrogen. The experiment is very similar to that of Thomson, but in this case a narrow pencil of X-rays was passed through a layer of the liquids and a photograph taken.

Clearly no explanation based on random crystal distribution is possible here. The explanation given was that of scattering at two centres separated by a distance, s, the directions of the line of centres being supposed to be distributed at random. On the classical theory of scattering this leads to a formula

$$I_\phi = A\left(1 + \frac{\sin x}{x}\right)$$

where I_ϕ measures the intensity along a direction ϕ with the original beam and A is a constant. The value of x depends on ϕ and is given by $x = \frac{4\pi s}{\lambda} \sin \frac{\phi}{2}$, λ being the wave-length of the incident radiation. We do not wish to go into the theory of this scattering here,[3] all that is necessary for us is to note that this formula leads to maximum values of I_ϕ along certain directions, for it is almost identical with that considered in Chapter II in connection with the diffraction of mechanical waves.

Maxima occur in fact for values of x approximately equal to $\frac{5}{2}\pi$, $\frac{9}{2}\pi$, etc., and actually at $x = 2{\cdot}459\pi$, $4{\cdot}477\pi$, etc., the half integer values becoming more and more exact as the numbers increase.

In the experiment on X-rays the diameters of the

[1] *Proc. Phys. Soc.*, B. L. Worsnop, June, 1928.
[2] *Jour. de Phys.*, 1923, p. 144.
[3] See *X-Rays and Electrons*, Compton, p. 384.

observed rings were measured and the corresponding values of ϕ deduced, so that s could be obtained. It was found that s corresponded to the distance between closely packed molecules and we may reasonably assume that the scattering centres in these liquids are molecules.

It is worth while to examine the cathode ray patterns by this means, especially for substances which would perhaps give no Debye-Sherrer pattern on account of non-crystalline structure.

From the mechanical wave-length, $\lambda = \frac{h}{\sqrt{2mPe}}$ and from the formula for s given above, writing

$$\sin \frac{1}{2}\phi = \frac{1}{2}\phi = \frac{1}{4}\frac{D}{L}$$

we find $\frac{D}{\lambda}$ = constant and consequently $D\sqrt{P}$ = constant.

Thus the constancy of $D\sqrt{P}$ is required by scattering also.

If we examine the pattern for the aluminium film in Thomson's experiment, assuming that the case is one of molecular scattering, the value of s works out at 2·22Å. The value of this quantity calculated for comparison by a formula used by Keesom and Smedt is 3·61, so that the agreement is better for the crystal reflection theory than for the theory of molecular scattering. It must not be forgotten though, that scattering might be from adjacent atoms. We do not wish to discuss the merits of one or other of these views, our point is merely that whatever view one takes it appears that electrons behave as waves.

Finally we mention another experiment, which is not yet completed, in which comparison of electrons with X-rays is very close.

It is known that when X-rays are diffracted by a grating with a large angle of incidence, nearly 90°, a diffracted pattern is obtained. If it be possible to

diffract a stream of electrons in a similar way, the result should indicate in a very direct manner the wave-like behaviour of electrons.

It will appear from this chapter that the new theory is pointing the way to experiment and that there is a possibility that these waves have an actual existence; further experiment must decide this for us.

CHAPTER V

THE PHYSICAL SIGNIFICANCE OF THE WAVE EQUATION AND THE WAVE FUNCTION

The Density and Probability Interpretation of ψ

IT is of the greatest interest and importance to inquire whether the wave function has a physical significance and to attempt to discover the true place of the wave equation in physics. It is not in the spirit of mathematical physics to introduce an isolated function and a differential equation into the theory and to leave it there without inquiring about its relation to the accepted doctrine.

We have always to begin with some fundamental assumption or, it may be, several assumptions, but we try to reduce the number to a minimum.

This has been illustrated in this book in the case of mechanics in our study of the Hamilton principle or in the example of the Hamilton-Jacobi equation. We may look upon Hamilton's principle, for example, as the fundamental assumption and from this deduce the classical mechanics. Such principles play an important part in all branches of physics and it does not surprise us that the wave-equation can be derived from a principle of this kind. We may have to be content with this and look upon the equation as an isolated principle with no connection with anything familiar to us.

We have developed a theory which leads us by analogy with optical theories to the wave equation and introduces a quantity, ψ, but, after all, we have

not actually built the equation into the classical theory, rather it lies upon it and ψ has not been associated with any physical quantity.

We propose in this chapter to review the work of embodying ψ and its equation into our physical doctrine.

First we consider Schroedinger's interpretation of ψ, which he adopts by analogy with the classical theory of radiation from a system of electrically charged particles. In this theory the radiation is determined by the electric moment of the system of particles.

This quantity is measured by reference to three co-ordinate axes, and if the charges have x-co-ordinates x_1, x_2, x_3, etc., and magnitudes e_1, e_2, e_3, etc., the x-component of the moment is given by :

$$p_1 = e_1x_1 + e_2x_2 + e_3x_3 + \text{etc.}$$

and there are similar y- and z-components.

If the charges oscillate the co-ordinates will be given as some function of the time and p_1 can be represented as a Fourier's series, which may be written thus :

$$p_1 = A_0 + A_1 \cos(2\pi\nu t + \delta_1) + A_2 \cos(4\pi\nu t + \delta_2) + \text{etc.} \quad (5\cdot1)$$

the general term being : $A_n \cos(2\pi n\nu t + \delta_n)$, ν being the fundamental frequency of the system.

It is more convenient to express the series in another way.

$$A_n \cos(2\pi n\nu t + \delta_n) = \tfrac{1}{2}(A_n e^{i\delta_n} e^{2\pi i n\nu t} + A_n e^{-i\delta_n} e^{-2\pi i n\nu t})$$

where $i = \sqrt{-1}$.

A_n is a real quantity and the only difference between $A_n e^{i\delta_n}$ and $A_n e^{-i\delta_n}$ is in the change of sign before the imaginary root. Quantities with this relation are said to be conjugates. We shall write C_n for the first and C_{-n} for the second, so that the general term may be written

$$C_n e^{2\pi i n\nu t} + C_{-n} e^{-2\pi i n\nu t}$$

and the value of p_1 is given by :

$$p_1 = \sum_{-\infty}^{\infty} C_n e^{2\pi i n\nu t} \quad . \quad . \quad . \quad . \quad (5\cdot2)$$

the summation being extended over all positive and negative integer values of n.

Thus the moment may be considered as equivalent to the super-position of harmonic oscillations of frequencies ν, 2ν, etc., the amplitudes of these oscillations being measured by A_1, A_2, etc., and the energies being proportional to A_1^2, A_2^2, etc.

If we use the expression (5·2) instead of (5·1) from which the A's have disappeared we shall prefer to express the energies or intensities of the corresponding component oscillations by C_1C_{-1}, C_2C_{-2}, etc.

Since we have $A_1^2 = 4C_1C_{-1}$ and in general $A_n^2 = 4C_nC_{-n}$ so that we measure the intensities conveniently by the product C_nC_{-n} to which they are proportional.

Since C_n and C_{-n} are complex conjugate quantities we shall adopt the usual notation : $C_nC_{-n} = |C_n|^2$.

The classical theory of radiation made the assumption that such a system would give rise to radiations of the frequencies ν, 2ν, etc., occurring in the series and that the intensities of the corresponding waves would be proportional to $|C_1|^2$, $|C_2|^2$, etc.

Bohr regarded the principle as essentially a quantum principle and showed that from the quantum principle one derived the classical principle when the quantum numbers, n, were large.

This principle of Bohr's, known as the correspondence principle, is evidently of a fundamental significance and Shroedinger recognizes this by adopting an expression corresponding to (5·2) to represent the state of a system in relation to the radiation it will emit.

The series he adopts is not a Fourier's series, though it is of the same type. Instead of using sine and cosine terms he makes use of characteristic functions and the associated characteristic values. Let these be ψ_1, ψ_2, etc., and E_1, E_2, etc.

Then it is assumed that the expression :

$$\psi = c_1\psi_1 e^{\frac{2\pi i}{h}E_1 t} + c_2\psi_2 e^{\frac{2\pi i}{h}E_2 t} + \text{etc.}$$

$$= \Sigma c_n\psi_n e^{\frac{2\pi i}{h}E_n t}$$

where the c's are constants, describes the radiation state.

The conjugate of ψ is written $\bar{\psi}$ and is evidently

$$\bar{\psi} = \Sigma c_n \bar{\psi}_n e^{-\frac{2\pi i}{h} E_n t}$$

Shroedinger makes the assumption that $\psi\bar{\psi}$ represents the density of charge and implies that the system is replaced by, or at any rate compared with, a continuous distribution of charge.

It may seem that this is very arbitrary and doubtless this is so, but it is in keeping with his theory, which is one of continuities.

This view then replaces the discrete charges with a continuous density and the electric moment is no longer to be written

$$p_1 = \Sigma e_n x_n$$

but is replaced by:

$$p_1 = \int \rho x dv.$$

or by

$$p_1 = \int \psi \bar{\psi} x dv$$

because of the assumption concerning $\psi\bar{\psi}$.

The product $\psi\bar{\psi}$ contains as a typical term:

$$c_m c_n (\psi_m \bar{\psi}_n e^{2\pi i \nu_{mn} t} + \psi_n \bar{\psi}_m e^{-2\pi i \nu_{mn} t})$$

where $\nu_{mn} = \dfrac{E_m - E_n}{h}$, and thus denotes the frequency of radiation corresponding to a transition between two states according to the view of the old quantum theory.

Thus a typical term in the expression for p_1 is:

$$c_m c_n e^{2\pi i \nu_{mn} t} \int \psi_m \bar{\psi}_n x dv + c_m c_n e^{-2\pi i \nu_{mn} t} \int \psi_n \bar{\psi}_m x dv.$$

The integrals are denoted by x_{mn} and x_{nm} respectively and we have:

from $x_{mn} = \int \psi_m \bar{\psi}_n x dv$ the result $\bar{x}_{mn} = \int \bar{\psi}_m \psi_n x dv = x_{nm}$.

$$\text{Thus } p_1 = \Sigma c_m c_n (x_{mn} e^{2\pi i \nu_{mn} t} + x_{nm} e^{-2\pi i \nu_{mn} t})$$
$$= \Sigma c_m c_n (x_{mn} e^{2\pi i \nu_{mn} t} + \bar{x}_{mn} e^{-2\pi i \nu_{mn} t}) \quad . \quad . \quad (5{\cdot}3)$$

In this we see an exact correspondence with the classical equation (5·2) where the coefficient associated with the exponential containing the negative power is the conjugate of that associated with the positive.

We thus replace the classical principle by (5·3) and suppose that the system gives rise to radiation of frequencies ν_{mn} and corresponding intensities proportional to $|x_{mn}|^2$.

These quantities, x_{mn}, are the components of the matrices introduced by Heisenberg in his discontinuous theory and the relations above show how these components are calculated from the characteristic functions.

We obtain from this a remarkable generalization of the classical and old quantum principles and must seek for an experimental verification.

An application has been made to the components of the lines H_α, H_β, H_γ and H_δ appearing in the Stark effect and the agreement is better than that obtained by Bohr and Kramers who based their calculations on the older principle.

There is another interpretation of ψ which is different from that we have just been considering; this is the interpretation of Bohr and Heisenberg and rests on a theory very different from that of Shroedinger.

It is assumed that problems on electrons are essentially of a statistical character, and the product $\psi\bar{\psi}$ measures a probability factor; for instance, if we think of an electron in space mapped out by the usual three co-ordinate axes the probability that the electron lies in the element $dx\,dy\,dz$ is $\psi\bar{\psi}\,dx\,dy\,dz$.

Since the electron is assumed to lie in a certain region of space it follows that the integral of this expression taken over the space is unity. This means that some

appropriate factor must be applied to ψ to make the integral $\int \psi\bar{\psi}dxdydz = 1$. This does not affect the fact that ψ is a solution of the wave equation and the ψ when determined to satisfy this integral relation as well is said to be normalized.

We have put forward in this book the view of Shroedinger, which is based on the assumption of continuity in nature and the methods adopted are those of classical physics.

There is, however, a different point of view which we may briefly mention since it is the basis of the interpretation of ψ as a probability function.

Bohr and Heisenberg state that there is a fundamental indefiniteness and a fundamental discontinuity associated with all quantities which we seek to observe in experimental physics. This is illustrated by an example which was given by Heisenberg, and although it is of a very special character there is nothing in the result which is a consequence of the special circumstances existing. The result is quite general.

An experiment is imagined for the determination of the position of an electron and it is shown that although we may conceivably locate the electron to any order of accuracy we please we cannot at the same time observe both its position and momentum without incurring an error. The more accurately we locate the electron the more inaccurately do we determine its momentum, and conversely, and if q_1 denote the error in determining the position and p_1 the error in the determination of the momentum, then the product p_1q_1 is of the order h. This, according to Bohr and Heisenberg, is the real significance of Planck's constant.

The special case considered by Heisenberg is that in which the electron is located by an optical method. We know that the error in determining the position of a particle in a microscope is of the order of the wavelength of the light used and it is necessary in the attempt

to observe very small objects to use light of very short wave length.

Two points cannot be resolved or recognized as distinct by means of a lens if they are closer together than $\frac{\lambda}{A}$, where λ is the wave-length and A the aperture or the angle subtended by the lens at either of the two points.

The error in location is then of the order $\frac{\lambda}{A}$.

Now let it be imagined that radiation is used to locate an electron in this way. The experiment and the apparatus are, of course, purely imaginary, but we will speak of the location of the electron by the gamma ray microscope, since gamma rays have the shortest known wave-lengths.

Then our quantity q_1, the error in our determination of the position of the electron, is of the order $\frac{\lambda}{A}$ and we will write :

$$q_1 \sim \frac{\lambda}{A}.$$

Now the rays which fall upon the electron give it momentum, we have in fact an example of the Compton effect and Compton's calculation in which he attributes to the radiation a momentum $\frac{h\nu}{c}$ is verified experimentally.

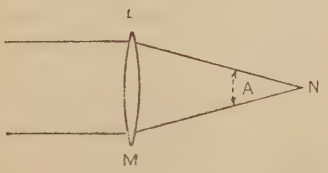

FIG. 17.

Part of this momentum is taken up by the electron so that in the act of locating it its momentum is changed.

Now the direction of the radiation lies within the angle A, and we do not know the direction any more accurately than this. At the worst we do not distinguish between the direction LN and MN along which the quanta may come.

Thus we do not distinguish between the momentum p along, say, AB and p along CB, and consequently make a mistake, which at the maximum is equal to AC and is thus of magnitude pA since A is a small angle.

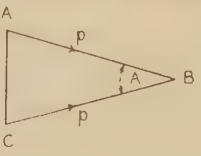

FIG. 18.

Thus the error in p is of the order pA and we write:

$$p_1 \sim p\mathrm{A}$$

or

$$p_1 \sim \frac{h\nu}{c}\mathrm{A}$$

ν being the frequency of the radiation and equal to $\frac{c}{\lambda}$ where c is the velocity of the radiation.

Thus

$$p_1 q_1 \sim h.$$

With this inherent uncertainty in our problems it is necessary to take a statistical view so that the introduction of a probability function is essential; this function, as we have seen, is regarded as $\psi\bar{\psi}$.

GEOMETRICAL AND METRICAL INTERPRETATIONS OF ψ

In the development of mathematical physics geometry has played a very great part. About Newton's time geometrical forms of argument were regarded as the most elegant, but to the physicist of to-day geometry has come to be more than a form of argument—more

even than the stage for the play of physical phenomena which it was to the classical physicist. Geometry is something physical, some phenomena we recognize as wholly geometrical, they are the expression of the geometry of space.

This new position of geometry in physics has been brought about, as is of course universally known, by Einstein. With this success of geometry in the realm of physics and with the elegance of its mathematical expression, it is natural to attempt further conquests and to try to include in a geometrical scheme all the phenomena of physics.

It does not appear possible to do this in four dimensions. Weyl and Eddington, however, made a very natural extension to the principle of relativity by assuming that the metrics of space had also physical significance, the phenomena corresponding to the conception of metrics being those of electromagnetism.

In this way it is possible to introduce by means of a system of geometry and metrics the right number of quantities for the description of the gravitational and electromagnetic phenomena.

The position at this stage is that the nature of space, both in regard to its geometry and metrics, is determined by the matter and electromagnetic phenomena present in it.

Gravitational phenomena correspond to the geometry, electromagnetic phenomena to the metrics.

The view taken by Weyl is that when a line-element is displaced parallel to itself in a space in which there is no electromagnetic phenomenon the length of the line remains unaltered, while a change occurs if such a phenomenon is present.

This may be expressed by saying that the line-element at a point in space is measured by ds where ds is determined by a quadratic expression :

$$(ds)^2 = \lambda^2 g_{mn} dx^m dx^n.$$

This corresponds to the familiar theorem of Pythagoras.

A summation is implied over m and n which may have the values 1 to 4, g_{mn} is a typical gravitational coefficient and λ is a metrical factor depending on the co-ordinates of the point which locates one end of the short line element ds.

It is implied that at each point there is a special scale-factor or metrical constant which determines the system of measurement. There is a relation amongst the coefficients, g_{mn}, which is the law of gravitation, a relation which cannot be deduced, it is of a fundamental character and must be postulated. The actual form of the relation is suggested by purely mathematical considerations, and while its determination is one of the triumphs of the appeal made to mathematical form the adoption of the equation is arbitrary.

Similarly we may anticipate an equation for λ. Possibly we may anticipate a single equation giving a relation between the g's and λ, giving us both the law of gravitation and the law of metrics; at any rate this would be a simplicity to be hoped for.

Now there are indications that the form of this equation in λ is that of 2·22, and we write tentatively

$$\text{div (grad } \lambda^2) = k\lambda^2$$

as our equation for the scale factor.

We must determine k from the consideration of actual problems and we shall omit any consideration of the value of k in general. Our chief point is to consider the suggestion that the wave equation is the fundamental equation of the metrics of space and ψ determines the scale factor.

For an electron in the absence of a gravitational and electromagnetic field the equation reduces to a simple form and the value of k to $-\frac{4\pi^2 m_0^2 c^2}{h^2}$ where m_0 is the rest mass of the electron.

Another very interesting consequence of this union of relativity and quantum mechanics is that the world-line of an electron is fundamentally indivisible. This

may be deduced from the above interpretation of λ or from a simpler consideration.

The world-line of any particle is obtained by assigning to the point where it is situated its three co-ordinates $(x\ y\ z)$ and a fourth for the time at which the particle is situated at the point.

It is convenient to call the four values $(x\ y\ z\ t)$ the co-ordinates of the particle in four-dimensional space and by analogy with three dimensions to imagine a curve drawn connecting the consecutive points; this curve is the world-line.

There is a convenience in describing the fourth point by a quantity u instead of t where $u = ict$. $(i = \sqrt{-1})$.

By way of illustration we will represent this by a two-dimensional diagram in x and u. We are thus considering a space of one dimension in length and one in time.

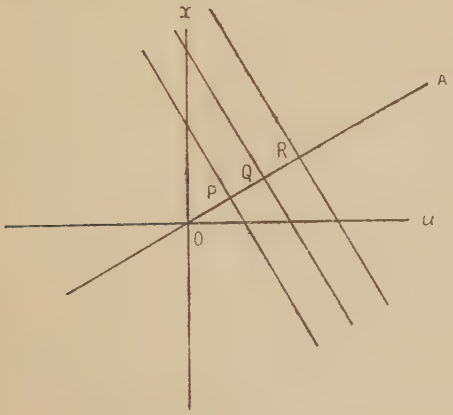

Fig. 19.

Suppose a particle is at rest at a point O, then its x-co-ordinate remains the same but as time progresses the u-co-ordinate increases and we obtain a series of

points on the u-axis. In this case the world-line is the u-axis.

If the particle is moving uniformly so that its x-co-ordinate increases in proportion to the time, the world-line is clearly a straight line, OA, inclined to both axes.

It appears in this way that to a point in motion, as we consider it in the usual way we have a line when time is treated as if it were a co-ordinate like the others. From our usual conception of motion in three dimensions we turn to something static in four or, as is illustrated in our diagram, a moving point becomes a fixed line.

The theory of relativity teaches that this is more than a convenient notation, it accepts the existence of a four-dimensional space of which the physicist at any instant appreciates a three-dimensional section.

Just as our moving point becomes a fixed line, so our moving three-dimensional surfaces of Shroedinger's theory become fixed four-dimensional surfaces.

These surfaces cut the world-line orthogonally and in the simple case we are considering we can associate with each point on the world line, OA, another line perpendicular to OA which represents the section of the surface in the plane considered. These surfaces are surfaces of constant action and we can associate them with the world-lines in the same way that we associate equipotential surfaces with lines of force. Three of these are drawn through P, Q and R in the diagram.

Now it is established in the quantum theory that action is never observed in an amount less than that measured by Planck's constant h.

It is therefore natural to draw our surfaces separated from one another in steps of this amount. For P the action is, say, H, for Q and R it is $H + h$ and $H + 2h$.

We have to consider the mechanical problem as if the waves and consequently the action surfaces have a real existence and since it is impossible to recognize physically that a surface lies between that at P and Q it is impossible to locate a point between P and Q,

so that the least distance we can determine by any physical means on the world-line is PQ. We must determine the distance PQ between two consecutive surfaces, the action values of which differ by h.

In four dimensions we have to consider a line element ds given by: $ds^2 = dx^2 + dy^2 + dz^2 + du^2$ or a more general quadratic expression.

In the case we are considering this simple form applies, since we have an electron with no gravitational or electromagnetic field.

It is convenient to introduce a quantity, τ, known as the proper-time which is such that $ds^2 = -c^2 d\tau^2$.

This proper-time measures the ordinary time when the particle is at rest, for then $dx = dy = dz = 0$ and $ds^2 = -c^2 d\tau^2 = du^2$, and if we remember that $du = icdt$, it follows that $d\tau = dt$.

Now the momentum in this space is a quantity defined by the components (p_x, p_y, p_z, p_u) with $p_x = m_0 \frac{dx}{d\tau}$, etc., from which it follows that the magnitude of this quantity, viz., $\sqrt{p_x^2 + p_y^2 + p_z^2 + p_u^2}$, is $\sqrt{-m_0^2 c^2}$ or $im_0 c$, which is, of course, a constant.

We introduce a quantity H related to the momentum by: $p_x = \frac{\partial H}{\partial x}$, etc., clearly H has the dimensions of momentum multiplied by length, i.e. of action and

$$\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2 + \left(\frac{\partial H}{\partial u}\right)^2 = -m_0^2 c^2,$$

by the momentum relation just considered. We may write this: $\left(\frac{dH}{ds}\right)^2 = -m_0^2 c^2$ which shows that when H changes by h, the change in s is given by $i\frac{h}{m_0 c}$. The occurrence of the imaginary need not trouble us, it has entered as a matter of convenience in the notation. This measures the smallest element of the world-line

we can measure in physics. If we determine the corresponding element of proper-time we can say that the minimum observable proper-time is $\frac{h}{m_0c^2}$, the mass m_0 referring to the fundamental particle considered, electron or proton.

It is perhaps more convenient to translate this in terms of ordinary space and time measurements. Let us consider an electron moving with a velocity, v. Then if the distance described in time dt be dl we have $dl = vdt$ and $dx^2 + dy^2 + dz^2 = dl^2 = v^2dt^2$.

$$\therefore ds^2 = (v^2 - c^2)dt^2$$

Now ds is never less than $\frac{ih}{m_0c}$ so that dt is never less than $\frac{h}{m_0c^2}\frac{1}{\sqrt{1-\beta^2}}$ and similarly dl is never less than $\frac{h}{m_0c}\frac{\beta}{\sqrt{1-\beta^2}}$, where $\beta = \frac{v}{c}$.

We can thus say that it is impossible to determine ordinary time and length intervals of less than these amounts. Thus there must be an inherent error in our time and length determinations. This suggests a similarity with the principle of Bohr and Heisenberg and we may examine the inherent error in the product p_1q_1 to compare with their result.

In measuring a length by any physical means we shall determine a number of fundamental units which we will denote by l_0 and write for our length nl_0. In this we recognize that there is a possible error of magnitude l_0. In measuring a time corresponding to this, as for example, in an experiment on an electron in motion, we shall determine n fundamental units and the time interval is nt_0, say, with a possible error t_0. Thus in determining a velocity we shall require to find $\frac{nl_0}{nt_0}$ which is of course equal to v but our determination gives us

$\frac{(n \pm 1)l_0}{(n \pm 1)t_0}$ and this in the worst case is $\left(1 \pm \frac{2}{n}\right)\frac{l_0}{t_0}$ with n large, as it is in practice, since the fundamental intervals are very small.

The error is thus of magnitude $\frac{2}{n}v$ since from the above values $\frac{l_0}{t_0}$ is v, the velocity required. We note that this error is large for observations on a few intervals.

The error in the location of position is l_0 so that our product p_1q_1 is $\frac{2}{n}l_0m_0v$ and on substituting the value of l_0 we find that this amounts to: $\frac{2h}{n}\frac{\beta^2}{\sqrt{1-\beta^2}}$ which is of the order h for a few fundamental lengths, and for the speeds occurring in atomic problems.

We note that according to this the error in the momentum is introduced in the determination of a length and a time, for according to the four-dimensional view the momentum is a constant, m_0c, which should be observed as a whole. This, however, would only be possible for a four-dimensional observer.

According to our view the four-dimensional observer is liable to an error in his length determination of an amount $\frac{h}{m_0c}$, so that if we take the measure of momentum as required in four dimensions the product p_1q_1 is still of order h.

This error is evidently due to some cause sufficiently fundamental to affect four-dimensional space, and is not due to the additional complications introduced in the conception of motion by the three-dimensional observer.

If we may apply these considerations to a quantum, treating it as a localized energy particle, then the error in locating it in space or in time becomes very large on account of the fact that $\beta = 1$ for a quantum.

This would mean that a quantum cannot be localized in space or time ; the momentum is, however, a definite quantity of magnitude m_0c, and with a mass obtained by dividing the energy $h\nu$ by c^2 we obtain the usual quantity $\frac{h\nu}{c}$. In four dimensions we meet with the same uncertainty in the world-line, if we may speak of the world line of a quantum. Speculative as this application to a quantum may be, the results agree with the state of our knowledge of light quanta.

There is another interesting result which may be derived from this principle of minimum proper-time. The shortest known intervals of time in mechanics are those occurring in atomic problems. If we consider the time of description of the inner orbit of an atom we may from the principle state that it must not be less than the specified amount. If we take the simplest case in which the inner orbit is circular and is described about a nucleus of order N, it is possible to determine the intervals in terms of N and certain other quantities. This interval is, of course, $\frac{2\pi r}{v}$, where r is the radius and v the velocity in the orbit.

The equation of motion is $\frac{mv^2}{r} = \frac{Ne^2}{r^2}$ with $m = \frac{m_0}{\sqrt{1-\beta^2}}$ and Bohr's momentum condition is $mvr = \frac{h}{2\pi}$.

We obtain $\frac{2\pi r}{v} = \frac{h}{m_0c^2}\frac{(1-\beta^2)^{\frac{1}{2}}}{\beta^2}$ which must be not less than $\frac{h}{m_0c^2}\frac{1}{(1-\beta^2)^{\frac{1}{2}}}$. From this we deduce that $\beta^2(1-\beta^2)^{\frac{1}{2}} > \frac{1}{2}$ or the orbital velocity cannot exceed $\frac{1}{\sqrt{2}}$ times the velocity of light.

This has nothing to do with the law of force in the atom, for we have made no use of the equation of motion.

From this equation we find $v = \frac{2\pi Ne^2}{h}$ whence $\frac{2\pi Ne^2}{h} \gtrless \frac{1}{\sqrt{2}}c$ or $N \gtrless \frac{hc}{2\sqrt{2}\pi e^2}$ which gives the highest nuclear order with an electron in the inner nucleus; thus the number of atoms constructed according to the Rutherford model is limited. If we work out the value of the quantity on the right-hand side we obtain approximately 97, the number being a little over 97, so that we expect the number to be limited to 97. This is close to the value for uranium, the extreme element, which has a nuclear number 92. Apparently the only determination of a theoretical character of the upper limit to the number of elements is a remarkable calculation by Sir J. H. Jeans based upon astronomical considerations. He obtains a number of about 92 or 93, but it is somewhat uncertain to what degree of accuracy this may be expected to be the actual maximum.

We must, of course, remember that our minimum time was obtained for a free electron, so that possibly some correction must come in for an electron in the field about the nucleus. Also it should be noted that no limitation is placed on the order of the nucleus itself, the limitation is on the orbits of electrons about the nucleus. There is no reason why an isolated nucleus stripped of the electron satellites should not attain any order, but if its order is greater than 97 then it will have no electrons in the deepest paths in the atom and will consequently not be an atom as ordinarily defined. It is interesting that the condition permits of higher order elements with empty inner rings; no such elements have been discovered.

If we apply the condition to an electronic path of order n we have to write $mvr = n\frac{h}{2\pi}$ and we find

$\beta^2 \not> \dfrac{1}{1+\dfrac{1}{n}}$, which shows that the farther out from the centre the electron lies the greater is its permitted velocity, which approaches c as the electron lies farther and farther out and becomes more and more nearly a free electron. This is, of course, exactly what is required by the view that the velocity of a particle has an upper limit in c.

Now returning to the attempt to correlate natural phenomena into a geometrical scheme, we must admit that so far the attempt has not been altogether satisfactory and not so fruitful as one hoped at first. Perhaps the brilliance and success of Einstein's theory made us expect too much, at any rate the metrical interpretation has not met with general acceptance. Einstein, who expressed doubts about it at first, appears never to have adopted it, and so far as may be gathered he has now rejected it. One must always remember that the acceptance or refusal of a theory may be ultimately a matter of taste ; this is especially so when there is no very decisive experimental evidence in favour of one rather than the other.

It is possible to attack this geometrical problem in another way ; this is, by introducing the conception of a five-dimensional universe.

The characteristic of this geometrical system is that the line element $d\sigma$ is measured by

$$d\sigma^2 = \mu_{mn} dx^m dx^n$$

where the coefficients, μ, correspond to the g's of four-dimensional space-time. We have, of course, more coefficients at our disposal, and these enable us to introduce the electro-magnetic potential into the geometrical quantities. The system has also another great advantage. We are familiar in Euclidean space with the conception that a particle under no forces moves along a straight line, and so in passing between two points it describes the least distance.

A line of minimum length joining two points is called a geodesic. Such lines are not always straight in the sense of Euclidean geometry; for example, if the line is limited to a spherical surface it is necessarily curved.

In the relativity theory of gravitation we accept the principle that gravitation is the physical expression that space is not Euclidean but is 'curved', bearing a relation to Euclidean space analogous to that which exists between a spherical and a plane surface.

A particle in a gravitational field then has to move in a non-Euclidean space, and the path it describes is a geodesic in this space. The particle is thus regarded as subject to no forces when it travels along the geodesic, for gravitational forces, as we have been accustomed to call them, no longer deviate the particle from its free path, which is the geodesic. We have been accustomed to look upon the Euclidean straight line as the free path and have regarded any deviation from that line as due to forces. In this way we banish the idea of a gravitational force.

This conception is very satisfactory from the mathematical point of view, because the motion of a particle is reduced to the statement:

$$\int ds = \text{a minimum}$$

$$\text{or } \delta \int ds = 0.$$

It is borne out by experiment in a very remarkable way, for it is equivalent to Newton's treatment but is more accurate and it explains certain things that Newton's laws did not.

But an electron in an electromagnetic field does not describe a geodesic; there is a deviation from this line, and we are obliged to keep the idea of electromagnetic forces. There seems no way out of this unless we admit a five-dimensional space; then it is possible so to determine the values of the coefficients,

μ_{mn}, that an electron in a gravitational and electro-magnetic field moves along the minimum path and the motion is summed up in the equation :

$$\delta \int d\sigma = 0.$$

It is difficult to imagine how the fifth dimension will become appreciable to us through physics. We have become accustomed to the idea of four dimensions and can regard this as a physical conception, the fourth dimension being experienced by three-dimensional observers through time. It does not at this stage appear possible for us to appreciate a fifth dimension in a similar way.

The limitations on the coefficients μ_{mn} which the geodesic principle imposes are of a very simple character, and it follows that the ordinary gravitational coefficients of space-time and the electro-magnetic potentials go to form the μ's in a very simple way. For example we have :

$$\mu_{mn} = g_{mn} - 2\chi\phi_m\phi_n' \text{ for } \frac{m}{n} = \Big\}1,\ 2,\ 3 \text{ or } 4$$

where χ is a constant and ϕ_m denotes a component of the electro-magnetic potential, while μ_{5m} = constant $\times\ \phi_m$ where $m = 1,\ 2,\ 3$ or 4. Another interesting thing is that the proper-time plays the same part in five dimensions as in four.

This reduction of motion under gravitational and electro-magnetic forces to a simple geometric conception is a great achievement and makes a great appeal on account of the simple mathematical form by which it is expressed.

The view we must take of this is that the physical four-dimensional universe is a part of a five-dimensional one, and that we are incapable of appreciating anything along the fifth dimension. If a particle at any instant leaves four-dimensional space it is lost to us completely, if it returns, then we have no know-

ledge of it at all between its exit and re-entrance. Doubtless it moves along a continuous path, but we only appreciate the particle discontinuously as it comes into four dimensions.

We may obtain an illustration of this in the following way. Let us imagine a being of one dimension capable at any instant of perception in that dimension only. Let us suppose that his line travels perpendicularly to its length and consequently sweeps out a plane. As the motion continues he will ultimately be able to explore the plane. The dimension perpendicular to the line in the plane will correspond to time, but he will not observe anything in a dimension normal to the plane at any time.

Let us suppose that a thread lies in a wavy curve which cuts this plane at equally spaced points. The observer will become conscious of this succession of points, and he may recognize a similarity between them ; he may say that he is observing the same object at consecutive instants. If the instants are very close together he may fail to recognize the points as separate, he will join them up by a world-line in the plane which he will call the world-line of the particle.

In any measurements he makes on the world-line he will naturally fail to detect any length less than that actually between the two points ; of course such a length is without significance, and it will be as if the world-line is made up of certain elements of a minimum observable length.

If we may use this analogy then the minimum length we have considered above may be of this nature ; it arises because four-dimensional calculations are made upon a five-dimensional continuous line.

According to what has been said the minimum separation between these points of exit and entry amounts to $\frac{h}{m_0 c}$ corresponding to a proper time $\frac{h}{m_0 c^2}$ and if we regard the proper-time as a five-dimensional quantity

we can say that the world line of the electron is a wavy five-dimensional line which comes into four dimensions at intervals of $\frac{h}{m_0 c^2}$ or with a frequency $\frac{m_0 c^2}{h}$. This frequency is that which de Broglie has ascribed to his so-called periodic phenomenon, and in the five-dimensional view of the universe this frequency finds a simple interpretation.

It can be shown that by introducing the function ψ it is possible to unite the laws of gravitation, electromagnetism and the wave-equation into a single five-dimensional variation principle. This is indeed a remarkable achievement.

The geometrical interpretation of physical quantities is very striking in this system, in which also the electronic charge is represented as a momentum corresponding to the fifth co-ordinate, for we find that to the four components p_1, p_2, p_3, p_4, which we have considered already, we must add a fifth, p_5, and our equations lead us to identify this with the charge e. But ψ in this system remains as a function introduced into a variational equation, and we would prefer that it should have a definite geometrical or metrical significance. We will bring our considerations to a close with a word or two on this point.

One suggestion that arises at once is that the five-dimensional universe may have a natural metric. This we obtain by analogy with the four-dimensional considerations which arose in connection with Weyl's theory. If we introduce a scale factor λ the wave equation may be regarded as giving λ. The wave equation can be reduced to a very simple form, for if we write $\frac{1}{\lambda^2}\frac{\partial \lambda^2}{\partial x^m} = a\,\pi_m$ it becomes $\operatorname{div} p_m = 0$, where the operation of divergence is referred to our new five-dimensional space. π_m itself has a very simple meaning, it is simply the extended momentum, i.e. it is a

component of the vector which corresponds to the ordinary mechanical momentum.

This idea of generalized momentum was introduced by Prof. W. Wilson in some relativity considerations a few years ago,[1] but its proper place in the theory is appreciated only by the adoption of a five-dimensional universe.

Wilson prefers to interpret ψ somewhat differently. He imagines a parallel displacement of a volume to take place in five dimensions, i.e. he thinks of a volume V bounded by its five edges, just as a pyramid is bounded in three dimensions by the three lines radiating from its apex, and each edge moves parallel to itself to a neighbouring point.

Associated with this displacement there is a volume change which is expressed as a ratio to the original volume, $\frac{\delta V}{V}$. This depends on the change in co-ordinates, and we write :

$$\frac{1}{V}\frac{\partial V}{\partial x^m} = K_m$$

We are now faced with circumstances exactly similar to those encountered in association with the vector potential of electro-magnetism. In this case it is usual to impose the condition that the divergence of the vector potential shall be zero. We find that by writing $K_m = a\,\pi_m$ where a is a constant such that $a^2 = -\frac{4\pi^2}{h^2}$, then $\operatorname{div} \pi_m = 0$ gives the wave equation, as of course in the preceding case, with the volume V as the variable and therefore proportional to ψ.

The view expressed by Wilson is that if a particle lies in a volume, V_0, at any instant, it will lie in the volume V obtained by parallel displacement from V_0 at some later or earlier instant.

The probability that a particle is situated in a given volume will depend in some way upon that volume,

[1] *Roy. Soc. Proc.* A, vol. 102, p. 478 (1922).

and so we come to the conception that ψ when interpreted in Wilson's way is a measure of probability, agreeing in this with the interpretation of Bohr and Heisenberg.

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